Research Statement

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My research lies in the field of low-dimensional topology, particularly knot theory and its applications towards the study of 3– and 4–manifolds. My papers and preprints may be found at `http://arxiv.org/a/ray_a_1`.

Low-dimensional topology is the branch of topology which studies manifolds of dimension four and lower. Techniques which have yielded much information about manifolds of dimension five and higher often fail for 3– and 4–manifolds, and in fact, many specialized tools are needed for studying these two particular cases. One may consider dimension four as a boundary case between low and high dimensions: there are enough dimensions for the manifold topology to exhibit complex behavior, but not enough space for our usual tools to work. This behavior is exemplified by the following: a closed manifold of dimension three or lower admits exactly one smooth structure; a closed manifold of dimension five or higher admits at most finitely many distinct smooth structures; however, a closed 4–manifold may have infinitely many distinct smooth structures.

A *link* is the image of a smooth embedding of a disjoint collection of circles into 3–space, considered up to isotopy; a *knot* is a link with a single component. The study of knots and links is intimately connected with the study of 3–manifolds as seen in the following famous theorem: any closed, connected, orientable 3–manifold can be obtained from the 3–sphere by performing a certain operation ('surgery') on some link. Just as the 3–dimensional relation of isotopy is related to the classification of 3–manifolds, there exist 4–dimensional relations on knots which are relevant to the classification of 4–manifolds.

My work so far has focused on these 4–dimensional equivalence relations, known as *concordance*. Knots under concordance form the *knot concordance group*, denoted by $\mathcal{C}$. In broad terms, my research aims to understand the structure of $\mathcal{C}$ using the following paradigms.

The action of satellite operators. A reasonable approach to studying any mathematical object is studying functions on it. In the case of knots, there is a natural choice of such functions, namely *satellite operators*, described in Figure 2. I study the action of satellite operators on the knot concordance group in [Ray13, DR13, CDR14, CR15, DR15, Ray15b, FR15]; this work is described in Section 1. Satellite operations are of independent interest beyond knot theory since they can be used to construct interesting examples of 3– and 4–manifolds. It is conjectured that satellite operators can be used to show that $\mathcal{C}$ is a *fractal* space. They also give *quasi-isometries* when $\mathcal{C}$ is considered as a metric space.

Filtrations of the knot concordance group. It is natural to seek to assess how 'close' a knot is to being trivial in $\mathcal{C}$, i.e. concordant to an unknotted circle. This notion was initially formalized when Cochran–Orr–Teichner introduced the *solvable filtration* of $\mathcal{C}$ and showed that the lower levels of the filtration encapsulate the information yielded by various classical concordance invariants—in a precise sense, the deeper a knot is within the solvable filtration, the closer it is to being trivial. Studying filtrations gives us a way of understanding the structure of $\mathcal{C}$, a large unwieldy object, in terms of smaller (and ideally simpler) pieces. There are several other filtrations of knot concordance. In [Ray15a] I define a new family of filtrations of $\mathcal{C}$ and establish relationships between these and various other filtrations; this is described in Section 2.

My ongoing and future projects, both within the above programs and otherwise, are described in Section 3. In my work so far I have used tools from geometric and algebraic topology, contact geometry, Heegaard–Floer homology, and other techniques.

1. Background

A *knot* is the image of a smooth oriented embedding $S^1 \hookrightarrow S^3$, considered up to isotopy. There is a natural operation on $\mathcal{K}$, the set of all knots, called *connected sum* which is shown below.

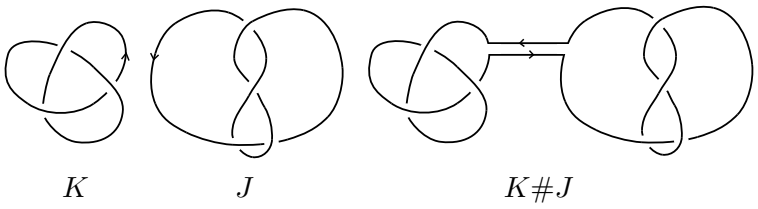

Figure 1. The connected sum operation on knots.

Two knots $K_0 \hookrightarrow S^3 \times \{0\}$ and $K_1 \hookrightarrow S^3 \times \{1\}$ are said to be *concordant* if they cobound a smooth, properly embedded annulus in $S^3 \times [0,1]$. Modulo concordance, $\mathcal{K}$ forms an abelian group under connected sum called the *knot concordance group*, denoted $\mathcal{C}$.

The connected sum operation can be generalized as follows. Given a knot P in a solid torus (called a *pattern*) and any knot K (called a *companion*), we obtain the *satellite knot* $P(K)$ by tying the solid torus into K in an untwisted manner, as shown in Figure 2. Any pattern P induces a function $P : \mathcal{C} \to \mathcal{C}$ taking $K \mapsto P(K)$ called a *satellite operator*. The number of times P wraps around the positively oriented longitude of the solid torus containing it is called the *winding number* of P.

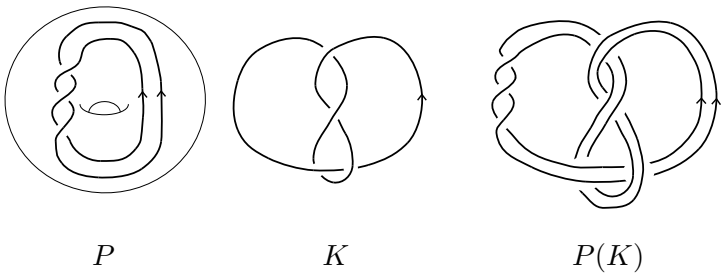

Figure 2. The satellite operation on knots. Here P is a pattern with winding number two.

We can also work entirely in the topological category; we say that two knots are *topologically concordant* if they cobound a proper, topologically embedded, locally flat annulus in $S^3 \times [0,1]$. Modulo topological concordance, $\mathcal{K}$ forms an abelian group called the *topological knot concordance group*, denoted $\mathcal{C}^{\text{top}}$. As before, any pattern P induces a satellite operator $P : \mathcal{C}^{\text{top}} \to \mathcal{C}^{\text{top}}$.

The knots in the class of the unknot, under either smooth or topological concordance, are important objects of study. A knot is called *(smoothly) slice* if it bounds a smooth, properly embedded disk in B^4, i.e. if it is concordant to the trivial knot. Similarly, a knot is *topologically slice* if it bounds a proper, topologically embedded, locally flat disk in B^4. There exist infinitely many knots which are topologically slice but not smoothly slice (see, for example, [End95, Gom86, HK12, HLR12, Hom14]).

A knot is said to be *ribbon* if it bounds an immersed disk in S^3 such that the singularities are of a particular type (called 'ribbon' singularities): it is easy to see that all ribbon knots are smoothly slice. It is a long-standing open question whether all slice knots are ribbon [Kir97, Problem 1.33].

2. The action of satellite operators

This section describes my completed work studying the action of satellite operators on the knot concordance group in rough chronological order. See Section 4 for ongoing and future projects.

2.1. **A non-orientable analogue of Kauffman's conjecture on slice knots.** Every knot K bounds an embedded, connected, compact, oriented surface in S^3. If a slice knot K bounds a punctured torus F then, up to isotopy and orientation, there are exactly two homologically essential simple closed curves J_1 and J_2 on F with zero self-linking [Gil83]. If either J_i is slice, we can construct a slice disk for K by "surgering" F along J_i, i.e. cut out a small neighborhood of J_i on F and glue two parallel copies of the slice disk for J_i to the remainder of F. Consequently, the curves J_i are called *surgery curves* for F. It can be seen that K is a certain satellite with ribbon pattern and companion J_i [CFT09]. In 1982, Kauffman conjectured that a knot with a genus one Seifert surface F is slice if and only if F has a slice surgery curve [Kau87, Strong Conjecture, pp. 226]. While there has been much evidence in the literature supporting this conjecture, such as in [CHL10, COT03, Coo82, Gil93], Cochran–Davis [CD15] have recently shown that Kauffman's conjecture is false.

In contrast, I showed that the natural non-orientable analogue of Kauffman's conjecture is true, as follows.

Theorem 2.1 ([Ray13])**.** *If a knot K bounds a punctured Klein bottle F with 'zero framing'—a linking number condition trivially satisfied in the orientable case—then there is a unique surgery curve J associated with F, K is a winding number two satellite with ribbon pattern and companion J, and K is slice in a $\mathbb{Z}\left[\frac{1}{2}\right]$–homology ball if and only if J is slice in a $\mathbb{Z}\left[\frac{1}{2}\right]$–homology ball. This implies that K is rationally slice if and only if J is.*

Being rationally slice is a strong condition; several concordance invariants obstruct knots from being rationally slice. For example, the Levine–Tristram signature function and Ozsváth–Szabó's τ-invariant [OS03] are both zero for rationally slice knots. Therefore, our result shows that, in marked contrast to the punctured torus case, there are very strong restrictions on the concordance class of surgery curves on punctured Klein bottles.

2.2. **Injectivity of satellite operators and a fractal structure on $\mathcal{C}$.** Theorem 2.1 has a surprising corollary about *cable* knots. The (p, q) cable of a knot K is obtained by applying the satellite operator induced by the (p, q) torus knot to K.

Corollary 2.2 ([Ray13])**.** *Given knots K and J and any odd integer q, the $(2, q)$ cables of K and J are concordant in a $\mathbb{Z}\left[\frac{1}{2}\right]$–homology $S^3 \times [0, 1]$ if and only if K is concordant to J in a $\mathbb{Z}\left[\frac{1}{2}\right]$–homology $S^3 \times [0, 1]$.*

The above is a result about the injectivity of cabling operations. Call a satellite operator *weakly injective* if $P(K) = P(U)$ implies $K = U$, and *injective* if $P(K) = P(J)$ implies $K = J$. Here U is the trivial knot and '=' denotes concordance in some category. A long-standing open question asks if the Whitehead doubling operator is weakly injective on $\mathcal{C}$ [Kir97, Problem 1.38]. In [CHL11], several 'robust doubling operators' were introduced and evidence was provided for their injectivity. However, Corollary 2.2 was the first complete result in the realm of injectivity, albeit in terms of concordance in $\mathbb{Z}\left[\frac{1}{2}\right]$–homology $S^3 \times [0, 1]$, i.e. '$\mathbb{Z}\left[\frac{1}{2}\right]$–concordance'. In [CDR14], we greatly generalized this result.

Theorem 2.3 ([CDR14, Theorem 5.1])**.** *Any strong winding number ± 1 satellite operator P is injective on $\mathcal{C}^{top}$. It is also injective on knots modulo concordance in a possibly exotic $S^3 \times [0, 1]$, which is the same as smooth concordance if the smooth 4–dimensional Poincaré Conjecture holds.*

Any winding number $n \neq 0$ satellite operator is injective on the group of $\mathbb{Z}\left[\frac{1}{n}\right]$–concordance classes of knots.

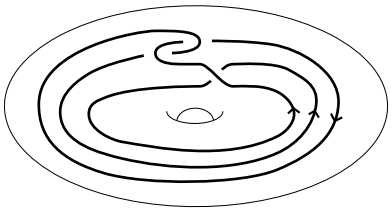

FIGURE 3. A strong winding number one pattern, denoted M. This pattern will be called the *Mazur pattern*.

For the sake of brevity, we omit discussion of '$\mathbb{Z}\left[\frac{1}{n}\right]$–concordance' and 'strong' winding number ± 1. It suffices to know that there exist multiple infinite families of strong winding number ± 1 patterns, and that any winding number ± 1 pattern which is unknotted as a knot in S^3 is strong winding number ± 1, such as the pattern shown in Figure 3. This particular pattern, denoted M, is called the *Mazur pattern*, since it appears in Mazur's first example of a contractible 4–manifold with boundary not homeomorphic to S^3 [Maz61]. Knots modulo concordance in a possibly exotic $S^3 \times [0,1]$ form the *exotic knot concordance group*, denoted $\mathcal{C}^{\text{ex}}$.

The question of injectivity of satellite operators is important in the study of the set of concordance classes of knots as a metric space in [CH14] where it was shown than winding number ± 1 satellite operators are quasi-isometries. Theorem 2.3 also provides strong evidence for a *fractal* structure on $\mathcal{C}$, conjectured in [CHL11]. We say a set has a fractal structure if there exist *self-similarities at arbitrarily small scales*, following [BGN03, Definition 3.1]. Theorem 2.3 shows that each strong winding number ± 1 satellite operator is a self-similarity for $\mathcal{C}^{\text{top}}$ as well as $\mathcal{C}^{\text{ex}}$; I addressed the question of scale in [Ray15b] via the following theorem.

Theorem 2.4 ([Ray15b])**.** *There exist infinitely many strong winding number one patterns P (such as the Mazur pattern in Figure 3) and a large class of knots K such that the knots $P^i(K)$ are distinct in $\mathcal{C}^{ex}$ and $\mathcal{C}$. That is, $P^i(K) \neq P^j(K)$ in $\mathcal{C}$ and $\mathcal{C}^{ex}$, for all $i \neq j \geq 0$.*

The action of the Mazur pattern of Figure 3 on $\mathcal{C}^{\text{ex}}$ may be compared to the action of $f(x) = \frac{x}{3}$ on the Cantor ternary set, in that iterations give distinct images of $\mathcal{C}^{\text{ex}}$ at smaller and smaller scales. To complete the fractal analogy one must also address the question of *surjectivity* of strong winding number ± 1 satellite operators, which we discuss in Section 2.4 below. One might also desire some notion of a metric to bolster the claim that $\mathcal{C}$ has a fractal structure. Some metrics on $\mathcal{C}$ were studied in [CH14]. I am interested in studying other metrics on $\mathcal{C}$ in the future.

2.3. **Distinct iterates of winding number one satellite operators.** Theorem 2.4 has several interesting applications. By choosing a topologically slice knot K, each pattern P in the theorem yields an infinite family $\{P^i(K)\}$ of smooth (and exotic) concordance classes of topologically slice knots. Several such examples already exist in the literature (see [End95, Gom86, HK12, Hom14]; ours are novel in the ease with which they are constructed and the added property that given any two knots in a family, one is a satellite of the other. Theorem 2.4 can also be used to construct infinite families of links with linking number one and unknotted components, which are each distinct from the class of the positive Hopf link. In a subsequent paper [DR15], Chris Davis and I used this theorem to construct an infinite family of links topologically, but not smoothly, concordant to the positive Hopf link; such examples were previous constructed by Cha–Kim–Ruberman–Strle in [CKRS12] but we show that our examples are distinct from theirs.

Theorem 2.4 is particularly interesting since it is generally hard to distinguish a knot K from a winding number ± 1 satellite $P(K)$ if $P(U)$ is unknotted (or slice). In this case, the 0–surgery manifolds – obtained by performed 0–framed surgery on S^3 along the knot – are homology cobordant and as a result K and $P(K)$ have the same classical concordance invariants, such as the Arf

invariant, the Levine–Tristram signatures, algebraic concordance invariants, etc. Theorem 2.4 circumvents these by utilizing the slice–Bennequin inequality from contact geometry and more recent smooth concordance invariants such as the τ and s invariants (this strategy was first used in this context in [CFHH13]). It is then interesting to ask whether these newer techniques can show that such satellite knots are *linearly independent* in $\mathcal{C}$ where classical invariants fail. The following result is in this vein.

Theorem 2.5 ([FR15])**.** *There exist infinitely many topologically slice knots K such that for any pattern P which can be changed to the trivial pattern by changing r positive crossings to negative crossings, and $\tau(P(K)) = \tau(K) + r$, $\{K, P(K)\}$ is linearly independent.*

Despite the fact that strong winding number ± 1 satellite operators are injective (Theorem 2.3), they do not necessarily map linearly independent subsets of $\mathcal{C}$ to linearly independent subsets as they are not in general homomorphisms. In the theorem below, we show that this does sometimes occur (see [FR15] for a more general statement).

Theorem 2.6 ([FR15])**.** *Let $\{K_n\}_{n=0}^{\infty}$ be a family of knots with arbitrarily small first singularity of the Upsilon function of* [OSS14]*. Let M denote the Mazur pattern shown in Figure 3. Then there is a subsequence $\{K_{n_l}\}_{n_l=0}^{\infty}$ such that $\{M(K_{n_l})\}_{n_l=0}^{\infty}$ is linearly independent.*

Let $\mathcal{T}$ denote the group of smooth concordance classes of topologically slice knots. As a corollary of the above theorem, we can infer that the image of the satellite operator $M : \mathcal{T} \to \mathcal{T}$ contains an infinite rank family of topologically slice knots which form the basis of a free summand of $\mathcal{T}$ (cf. the recent result of Adam Levine [Lev14] that M is not surjective on $\mathcal{T}$).

2.4. **The group of generalized patterns, and surjectivity of satellite operators.** The set of patterns has the structure of a monoid where the identity element is the trivial (winding number one) pattern given by the core of a solid torus, and the classical satellite construction is given by a monoid action. It is straightforward to see that patterns do not form a group [DR13, Proposition 2.1]. In [DR13], Chris Davis and I observe that the monoid of strong winding number ± 1 patterns has a natural inclusion into the set of homology cylinders, which form a group under homology cobordism. This group was introduced by Levine in [Lev01]. We show that homology cobordism classes of homology cylinders have a well-defined group action on the set of concordance classes of knots in homology 3–spheres, and that this action restricts to the usual satellite operation on $\mathcal{C}^{\text{top}}$ and $\mathcal{C}^{\text{ex}}$. In other words, we proved a theorem of the following type.

Theorem 2.7 ([DR13])**.** *Let $* = $ top or ex. Let $\widehat{\mathcal{C}^*}$ denote the group of concordance classes of knots in homology 3–spheres in the $*$–category. There is an obvious map $\Psi : \mathcal{C}^* \hookrightarrow \widehat{\mathcal{C}^*}$. Let $\mathcal{S}$ denote the monoid of strong winding number ± 1 patterns. There is a group $\widehat{\mathcal{S}^*}$ consisting of homology cylinders up to an appropriate notion of homology cobordism, and a monoid morphism $E : \mathcal{S} \to \widehat{\mathcal{S}^*}$. There is an action of $\widehat{\mathcal{S}^*}$ on $\widehat{\mathcal{C}^*}$ which restricts to the classical satellite construction on $\mathcal{C}^*$, that is, for any strong winding number ± 1 pattern P, the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{C}^* & \xrightarrow{P} & \mathcal{C}^* \\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle\Psi} \\ \widehat{\mathcal{C}^*} & \xrightarrow{E(P)} & \widehat{\mathcal{C}^*} \end{array}$$

Since $E(P)$ is an element of a group acting on $\widehat{\mathcal{C}^*}$, $E(P) : \widehat{\mathcal{C}^*} \to \widehat{\mathcal{C}^*}$ is a bijection.

This leads to a number of interesting results. For example, we give a complete characterization of patterns inducing surjective satellite operators [DR13, Proposition 3.2] and use this to show

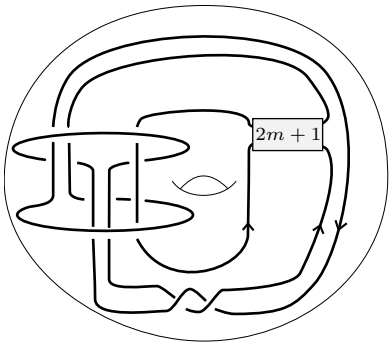

FIGURE 4. A family of patterns ($m \geq 0$) inducing non-trivial bijective satellite operators on $\mathcal{C}^{\text{top}}$ and $\mathcal{C}^{\text{ex}}$. (The box represents $2m+1$ negative half-twists.)

that there exists an infinite family of strong winding number ± 1 patterns, distinct from connected-sum patterns, which induce *bijections* on $\mathcal{C}^*$, such as those in Figure 4 [DR13, Corollary 3.7] (compare with the result of Adam Levine [Lev14] that there exist strong winding number ± 1 satellite operators that are far from being surjective on $\mathcal{C}^{\text{ex}}$). For a large class of patterns P, we are also able to explicitly draw patterns $\overline{P}$ such that $\overline{P}(P(K))$ is concordant to K for each knot K [DR13, Theorem 3.4, Proposition 3.5]. We are also able to easily reprove Theorem 2.3 via a simple diagram chase.

We remark in passing that [DR13] also also proves analogues of Theorem 2.7 for patterns with other non-zero winding numbers—including patterns of winding number ± 1 which are not strong winding number ± 1—but we omit them for brevity.

2.5. **Shake slice and shake concordant knots.** We have seen earlier that a knot K and its satellite $P(K)$ share a number of classical concordance invariants when P is a winding number one pattern with $P(U)$ slice. What is the equivalence relation generated by concordance as well as setting $K \sim P(K)$ for any winding number one pattern P with $P(U)$ slice? In [CR15], Tim Cochran and I show that this equivalence relation is the same as a natural generalization of concordance which we call *shake concordance*. For any knot K, define an *algebraically one collection* to be a collection of $2n+1$ 0–framed parallels of K, where $n+1$ of the parallels are oriented in the direction of K and the n remaining parallels are oriented in the opposite direction, for some $n \geq 0$. Knots K_0 and K_1 are said to be *shake concordant* if there is a smooth, properly embedded, compact, connected, genus zero surface A in $S^3 \times [0,1]$, where $A \cap S^3 \times \{0\}$ is an algebraically one collection of K_0 and $A \cap S^3 \times \{1\}$ is an algebraically one collection of K_1. This is clearly a generalization of knot concordance. Moreover, this is a relative version of *shake sliceness* of knots, defined by Akbulut in [Akb77]: a knot K is *shake slice* if an algebraically one collection of K bounds a smooth, properly embedded, compact, connected genus zero surface in B^4.

There are no known examples of knots that are shake slice but not slice. In [CR15] we give the first examples of knots that are shake concordant but not concordant; there are infinitely many examples, which can be chosen to be topologically slice. We also give a complete characterization of shake concordance in terms of concordance and winding number one satellites, as follows.

Theorem 2.8 ([CR15])**.** *Two knots K and J are shake concordant if and only if there exist winding number one patterns P and Q with $P(U)$ and $Q(U)$ ribbon, such that $P(K)$ is concordant to $Q(J)$.*

This yields the following corollary (compare Theorem 2.3).

Corollary 2.9 ([CR15])**.** *The equivalence relation on the set of knots generated by concordance together with the relation $K \sim P(K)$ for all K and all winding number one patterns P with $P(U)$ ribbon is the same as the equivalence relation generated by shake concordance.*

We also give a characterization of shake slice knots in terms of winding number one satellites and show the following.

Corollary 2.10 ([CR15])**.** *There exists a shake slice knot that is not slice if and only if there exists some winding number one satellite operator $P : \mathcal{C} \to \mathcal{C}$ with $P(U)$ ribbon which fails to be weakly injective.*

3. Filtrations of the knot concordance group

The *solvable filtration* $\{\mathcal{F}_n\}_{n=0}^{\infty}$ of $\mathcal{C}$ given in [COT03] has been instrumental in the study of (smooth and topological) knot concordance in recent years, particularly since the lower levels of the filtration encapsulate the information one can obtain from several classical concordance invariants. Part of the justification for the naturality of the solvable filtration is its close relationships with several more geometric filtrations of $\mathcal{C}$. For example, if a knot K bounds a grope of height $n+2$, or a Whitney tower of height $n+2$, then $K \in \mathcal{F}_n$ [COT03].

Cochran–Harvey–Horn [CHH13] have recently introduced a new pair of filtrations of $\mathcal{C}$, the *positive* and *negative* filtrations: $\{\mathcal{P}_n\}_{n=0}^{\infty}$ and $\{\mathcal{N}_n\}_{n=0}^{\infty}$ respectively. These new filtrations are of interest because (unlike the solvable filtration) they can be used to study smooth concordance classes of topologically slice knots. In [Ray15a], I give geometric analogues for these new filtrations in terms of *Casson towers* [Cas86, Fre82], certain 4–dimensional objects built using disks with transverse self-intersections (a schematic picture is given in Figure 5).

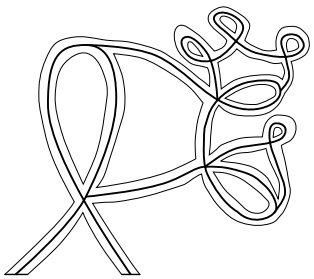

Figure 5. Schematic diagram of a Casson tower of height three.

If a knot K bounds an immersed disk in B^4 with only positive (resp. negative) self-intersections, $K \in \mathcal{P}_0$ (resp. $K \in \mathcal{N}_0$). Thus, Casson towers—built using layers of immersed disks—are natural objects to study in the context of the positive and negative filtrations. Let $\{\mathcal{G}_n\}_{n=1}^{\infty}$ denote the grope filtration of $\mathcal{C}$. In [Ray15a], I defined the filtrations $\{\mathfrak{C}_n\}_{n=1}^{\infty}$, $\{\mathfrak{C}_n^+\}_{n=1}^{\infty}$, $\{\mathfrak{C}_n^-\}_{n=1}^{\infty}$, $\{\mathfrak{C}_{2,k}\}_{k=0}^{\infty}$, $\{\mathfrak{C}_{2,k}^+\}_{k=0}^{\infty}$, and $\{\mathfrak{C}_{2,k}^-\}_{k=0}^{\infty}$, and established the following relationships.

Theorem 3.1 ([Ray15a, Theorem A])**.** *For any $n \geq 0$,*

(i) $\mathfrak{C}_{n+2} \subseteq \mathcal{G}_{n+2} \subseteq \mathcal{F}_n$,
(ii) $\mathfrak{C}_{2,\,n} \subseteq \mathcal{F}_n$,
(iii) $\mathfrak{C}_{n+2}^+ \subseteq \mathfrak{C}_{2,\,n}^+ \subseteq \mathcal{P}_n$,
(iv) $\mathfrak{C}_{n+2}^- \subseteq \mathfrak{C}_{2,\,n}^- \subseteq \mathcal{N}_n$.

The second inclusion in part (i) was mentioned earlier [COT03, Theorem 8.11]; we include it here for completeness.

In fact, we show that $\mathfrak{C}_3 \subseteq \bigcap \mathcal{F}_n$, and thus infer that either every knot in $\mathfrak{C}_3$ is topologically slice or there exist knots in $\bigcap \mathcal{F}_n$ which are not topologically slice. The only presently known elements of $\bigcap \mathcal{F}_n$ are topologically slice and it is an open question whether *all* knots in $\bigcap \mathcal{F}_n$ are topologically slice. While any topologically slice knot K bounds an arbitrarily tall Casson tower,

not all of them are even in $\mathfrak{C}_1^{\pm}$. For links of two or more components, each level of the positive and negative filtrations is non-trivial [CP14]. Mirroring the fact that the positive and negative filtrations non-trivially filter smooth concordance classes of topologically slice knots and links, it is expected that the filtrations $\{\mathfrak{C}_{2,n}^{\pm}\}_{n=0}^{\infty}$ will as well.

4. Ongoing and future work

I have several ongoing projects within the programs mentioned above, as well as some planned future work in different contexts.

Project 1. Linear independence of iterated winding number one satellites

We saw earlier that for a knot K, if $P(K)$ is a winding number one satellite under a pattern P with $P(U)$ slice, then K and $P(K)$ share several classical concordance invariants. I showed in [Ray15b] that in some cases the iterated knots $\{P^i(K)\}$ are distinct. I would like to investigate when such a family of iterated satellites (possibly when K is topologically slice) is linearly independent in $\mathcal{C}$. This project is in collaboration with Peter Feller, and led to our work in [FR15]. At the moment we are primarily investigating the Upsilon function of [OSS14] in this regard.

Project 2. Casson tower filtrations of $\mathcal{C}$

Several open questions remain about the Casson tower filtrations of $\mathcal{C}$ mentioned in Section 3. In particular, I would like to investigate whether the $\{\mathfrak{C}_{2,n}^{\pm}\}_{n=0}^{\infty}$ filtrations can distinguish between smooth concordance classes of topologically slice knots as expected. There is also a need to find examples of knots in arbitrarily deep levels of the filtrations. Once such examples have been found, it would be interesting to ask how large the successive quotients of the filtrations are, e.g. does each successive quotient contain a $\mathbb{Z}^{\infty}$?

Project 3. Shake slice knots that are not slice

In [CR15], Tim Cochran and I gave a characterization of shake slice knots in terms of winding number one satellites , namely that a knot K is shake slice if and only if there exists a winding number one pattern P with $P(U)$ ribbon such that $P(K)$ is slice. This suggests a novel approach to the long-standing open question of whether there exist shake slice knots that are not slice [Kir97, Problem 1.41(A)] (see Corollary 2.10).

Project 4. Structure in the double concordance group

A knot is called *doubly slice* if there exists a smooth unknotted 2–sphere Σ in S^4 such that $\Sigma \cap S^3 = K$. Similarly, K is *topologically doubly slice* if such a Σ is merely topologically locally flat. (If knotted 2–spheres are used, we recover the usual notion of slice and topologically slice knots (or possibly links)). In [Mei15], Meier gave infinitely many examples of slice knots that are topologically doubly slice but not smoothly doubly slice. Similar to how concordance can be defined using slice knots, there is a notion of double concordance of knots, leading to a group called the *double concordance group*, denoted $\mathcal{C}_{\mathcal{D}}$. In [Kim06], Kim gave a solvable bi-filtration of $\mathcal{C}_{\mathcal{D}}$, analogous to the solvable filtration for $\mathcal{C}$. In joint work with Peter Horn, we define a grope bi-filtration of $\mathcal{C}_{\mathcal{D}}$. We will determine the relationship between the grope and solvable bi-filtrations. Subsequently, we plan on defining analogues for the positive, negative, and bipolar filtrations for $\mathcal{C}_{\mathcal{D}}$. As in the case for $\mathcal{C}$ we hope that these filtrations will distinguish between knots that are topologically, but not smoothly, doubly slice.

Project 5. Embedded tori in contractible 4–manifolds

The classical loop theorem of Papakyriakopoulos [Pap57] states that if M is a 3–manifold and there is a map $f : (D^2, \partial D^2) \to (M, \partial M)$ such that $f|_{\partial D^2}$ is not nullhomotopic in ∂M, then there is an embedding with the same restriction on ∂D^2. In other words, an essential curve in ∂M which

extends to a map of a disk in M, extends to a embedding of a disk in M. This project asks whether there is an analogue of this theorem for tori in 4–manifolds, as follows. Let X be a contractible manifold with $\partial X = M$. Then given any embedding of a torus T into M, the map can be extended to a map from a solid torus. In joint work with Danny Ruberman, we are studying when we can find an embedded solid torus in X bounded by T. Additionally, we are interested in whether there are examples where an embedding of a solid torus exists in the topological category but not in the smooth category.

Project 6. Reversibility of knots in concordance

Given a knot K, its reverse, denoted rK, is obtained by reversing the orientation of the knot. In general it is hard to distinguish a knot from its reverse in concordance: most concordance invariants do not see the knot orientation. Examples of knots not concordant to their reverse were given in [Liv83, Nai96, KL99, Tam99, HKL10, CKL15] – in general, these showed that certain specific knots are not concordant to their reverses by computating either Casson–Gordon invariants or a twisted Alexander polynomial, as opposed to constructing infinite families of such knots. We hope to construct such a family using Theorem 2.3, which shows that if the (p, q)–cable of a knot K is concordant to its reverse, then K is '$\mathbb{Z}\left[\frac{1}{p}\right]$–concordant' to its reverse. Therefore, if any of the above referenced examples of knots fail to be rationally concordant to their reverse, then their cables form an infinite family of knots that fail to be concordant to their reverse. This would require understanding whether the obstructions from Casson–Gordon invariants or twisted Alexander polynomials also obstruct rational concordance, which is an interesting question in its own right.

Project 7. The group of generalized patterns and its group action on knots in homology spheres

As we described in Section 2, recasting the classical satellite operation as a restriction of a group acton has already yielded several interesting results, and we believe that further insights can be gained from this perspective. Jointly with Chris Davis, we are continuing the study of generalized patterns. This consists on one hand of investigating the group-theoretic properties of generalized patterns (e.g. is the group abelian? does it have any torsion? do connected-sum operators form a normal subgroup?) and on the other hand investigating the group action on concordance classes of knots in homology spheres (e.g. is the action free? faithful?)

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