Research Statement

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My research is in the field of low-dimensional topology, particularly the study of knots and 3– and 4–manifolds. My papers and preprints are available at http://arxiv.org/a/ray_a_1 and my profile on Mathscinet is located at http://www.ams.org/mathscinet/MRAuthorID/1039665.

Low-dimensional topology is concerned with manifolds of dimension four and lower. Standard high-dimensional techniques often fail for 3– and 4–manifolds and many specialized tools are required for these two particular cases. In some sense, dimension four is the boundary case between low and high dimensions; there are enough dimensions for the manifold topology to exhibit complex behavior, but not enough space for our usual tools to work. This phenomenon is exemplified by the following: a closed manifold of dimension three or lower admits exactly one smooth structure; a closed manifold of dimension five or higher admits at most finitely many distinct smooth structures; however, there exist closed 4–manifolds with infinitely many distinct smooth structures.

A *knot* is a smooth embedding of a circle in 3–space, considered up to isotopy. Knot theory is intimately connected with the study of 3–manifolds since any closed, connected, orientable 3–manifold can be obtained from the 3–sphere by performing a certain operation ('surgery') on a collection of knots. Just as the 3–dimensional relation of isotopy is related to the classification of 3–manifolds, there are 4–dimensional relations on knots relevant to the classification of 4–manifolds.

I study these 4–dimensional equivalence relations, known as *concordance*. Modulo concordance and under the connected sum operation, knots form the *knot concordance group*, denoted by $\mathcal{C}$. In broad terms, my goal is to understand the structure of $\mathcal{C}$, using techniques from geometric and algebraic topology, contact and symplectic geometry, Heegaard–Floer homology, etc.

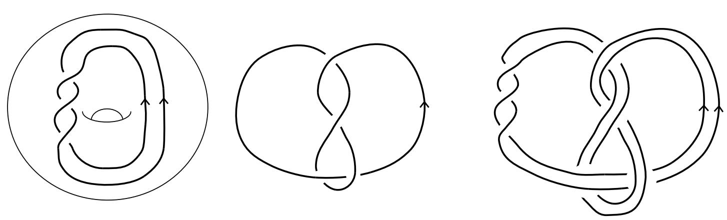

Figure 1. The satellite operation on knots.

A principal theme in my research is the study of a natural family of functions on $\mathcal{C}$, called *satellite operators*. Satellite operators are not group homomorphisms in general. In particular, I have studied the injectivity and surjectivity of these functions, the behavior of linearly independent subsets, and the dynamics under iteration. Aside from the group structure, I am interested in other means of organizing knot concordance classes, particularly in the form of filtrations and metrics derived from 4–dimensional geometric constructions. Studying $\mathcal{C}$ as a metric space, along with the injectivity of families of satellite operators, provides strong evidence that $\mathcal{C}$ is endowed with a fractal structure. I describe this further in Sections 1 and 2.

Certain special families of knot concordance classes warrant particular attention. For instance, I study smooth concordance classes of topologically slice knots, which provide a microcosm within which to understand the distinction between smooth and topological 4–manifolds. I explain this work in greater detail in Section 3. In Section 4, I describe a recent project where we apply techniques from knot concordance to show that a natural 4–dimensional analogue of Dehn's lemma does not hold. Section 5 consists of brief outlines of some ongoing and future work.

We say that two knots K_0 and K_1 are *concordant* if they cobound a smoothly embedded annulus in $S^3 \times [0,1]$. Modulo concordance, knots form an abelian group under the connected sum operation, called the *knot concordance group*, denoted $\mathcal{C}$. If we instead ask for locally flat annuli, we obtain the *topological knot concordance group*, denoted $\mathcal{C}_{\text{top}}$. Similarly, we may ask for smooth annuli in a possibly exotic copy of $S^3 \times I$, to obtain the *exotic knot concordance group*, denoted $\mathcal{C}_{\text{ex}}$. If the smooth 4–dimensional Poincaré Conjecture is true, then $\mathcal{C}_{\text{ex}} = \mathcal{C}$ [CDR14].

The knots in the class of the unknot, in any of the above categories, are of particular interest. A knot is called *slice* if it bounds a smoothly embedded disk in B^4, i.e. if it is concordant to the trivial knot. Similarly, a knot is *topologically slice* if it bounds an embedded locally flat disk in B^4. There are infinitely many topologically slice knots that are not smoothly slice [End95, Gom86, HK12, HLR12, Hom14]), and any such knot gives rise to an exotic copy of $\mathbb{R}^4$ [GS99, Exercise 9.4.23].

The *solvable filtration* $\cdots \mathcal{F}_{n.5} \subseteq \mathcal{F}_n \subseteq \cdots \mathcal{F}_{0.5} \subseteq \mathcal{F}_0 \subseteq \mathcal{C}$ [COT03] has been instrumental in organizing the study of knot concordance in recent years. A remarkable feature of the filtration is that lower levels encapsulate several classical concordance invariants. For example, a knot K is in $\mathcal{F}_0$ if and only if $\operatorname{Arf}(K) = 0$; similarly, $K \in \mathcal{F}_{0.5}$ if and only if K is algebraically slice. Moreover, the filtration is highly non-trivial – specifically, each $\mathcal{F}_n/\mathcal{F}_{n.5}$ has a subgroup isomorphic to $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty$ [CHL09, CHL11]. Additionally, if K is topologically slice, $K \in \bigcap_{n=0}^\infty \mathcal{F}_n$; however, the converse is open.

1. The action of satellite operators on $\mathcal{C}$

A *pattern* P is a knot in a solid torus V; the *satellite knot* $P(K)$ is obtained by tying the solid torus V into the knot K, in an appropriately untwisted manner, as shown in Figure 1. This operation is compatible with concordance, and thus descends to yield well-defined functions $P : \mathcal{C} \to \mathcal{C}$, $P : \mathcal{C}_{\text{ex}} \to \mathcal{C}_{\text{ex}}$, and $P : \mathcal{C}_{\text{top}} \to \mathcal{C}_{\text{top}}$, called *satellite operators*. The algebraic intersection number of P with a generic meridional disk of V is called the *winding number* of P.

Satellite knots are interesting both within and beyond knot theory. For instance, satellite operations can be used to construct distinct knot concordance classes which are hard to distinguish using classical invariants [CHL11, COT04]. They were used in [Har08] to subtly modify a 3–manifold without affecting its homology type. Winding number ± 1 satellite operators in particular are related to Mazur 4–manifolds [AK79] and Akbulut corks [Akb91]. As a result, there has been considerable interest in understanding the action of satellite operators on $\mathcal{C}$, e.g. in the famous conjecture [Kir97, Problem 1.38] that the Whitehead double of a knot K is smoothly slice if and only if K is smoothly slice. The Whitehead double is the simplest example of a non-trivial winding number zero satellite. All Whitehead doubles are topologically slice [Fre82], whereas they are not all smoothly slice [CG88]. Consequently, this conjecture can be seen as highlighting the remarkable difference between the topological and smooth categories for 4–manifolds. The Whitehead double of the unknot is slice. Thus, the conjecture above consists of a special case of the following question.

Question 1. Are non-trivial satellite operators injective?

That is, for a non-trivial satellite operator P and knots K and J, if $P(K)$ is concordant to $P(J)$, does this imply that K is concordant to J? A survey of recent work on the injectivity of Whitehead doubling may be found in [HK12]. In [CHL11], several winding number zero 'robust doubling operators' were introduced and evidence was provided for their injectivity. Not much else is known in the winding number zero case. In contrast, in joint work with Cochran and Davis [CDR14], we establish the following result for all non-zero winding numbers.

Theorem 2 ([Ray13, CDR14])**.** *All 'strong winding number ± 1' satellite operators are injective on $\mathcal{C}_{top}$ and $\mathcal{C}_{ex}$.*

In general, all satellite operators with non-zero winding number are injective modulo certain homological versions of concordance.

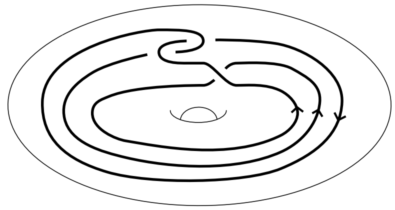

Figure 2. A strong winding number one pattern, denoted M. This pattern will be called the *Mazur pattern*.

'Strong' winding number is a homotopy analogue for the notion of winding number. In particular, the trivial knot can be embedded in a solid torus in infinitely many ways with winding number ± 1, such as in Figure 2; each such pattern is strong winding number ± 1. There are several other infinite families of strong winding number ± 1 satellite operators.

In some sense, the above theorem was generalized in [CH14], where it was shown that winding number ± 1 satellite operators are quasi-isometries with respect to certain natural metrics on $\mathcal{C}$. Our result also provides strong evidence for a *fractal* structure on $\mathcal{C}$, conjectured in [CHL11]. We say a set has a fractal structure if there exist *self-similarities at arbitrarily small scales*, following [BGN03, Definition 3.1]. Our result shows that infinite classes of winding number ± 1 satellite operators are self-similarities for $\mathcal{C}_{\text{top}}$ and $\mathcal{C}_{\text{ex}}$. The question of 'scale' is related to the dynamics of iterated satellite operators, which I address in the following theorem.

Theorem 3 ([Ray15b]). *For infinitely many strong winding number ± 1 satellite operators P, including the Mazur satellite operator from Figure 2, infinitely many knots K have non-periodic orbits, i.e. $P^i(K) = P^j(K)$ in $\mathcal{C}_{ex}$ (and thus in $\mathcal{C}$) if and only if $i = j$.*

The action of the Mazur pattern of Figure 2 on $\mathcal{C}$ may be compared to the action of $f(x) = \frac{x}{3}$ on the Cantor ternary set, in that iterations give distinct images of $\mathcal{C}$ at smaller and smaller scales. Moreover, A. Levine showed in [Lev14] that each iterate of the Mazur satellite operator (i.e. each iterated function M^i for $i \geq 1$) is non-surjective on $\mathcal{C}$. In contrast, in joint work with Feller and J. Park [FPR16], we show that the image of each iterate of the Mazur satellite operator is still 'large', even when restricted to smooth concordance classes of topologically slice knots, as follows.

Theorem 4 ([FPR16]). *There is an infinite family of topologically slice knots such that its image under any iterate of the Mazur satellite operator generates an infinite rank subgroup of $\mathcal{C}$.*

To complete the fractal analogy, there is a need for a well-defined notion of 'size' in $\mathcal{C}$. There has been recent work in this realm by Cochran–Harvey [CH14], using the slice genus of knots, and Cochran–Harvey–Powell [CHP15], using a refinement of the slice genus using gropes. In ongoing joint work with Cochran, Harvey, and Powell, we study a novel metric on knot concordance classes using certain 4–dimensional tower constructions. The proposed metric improves on the previous work due to being highly compatible with winding number one satellite operations (namely, the putative self-similarities) as well as various non-trivial filtrations of smooth concordance classes of topologically slice knots.

As we noted above, A. Levine [Lev14] showed that the Mazur satellite operator is non-surjective on $\mathcal{C}$. In contrast, in joint work with Davis [DR16], we show that there exist infinitely many non-trivial bijective satellite operators. We do so by recasting the satellite operation as a restriction of a natural group action as follows. The set of patterns has the structure of a monoid where the identity element is the core of a solid torus and the satellite construction is a monoid action. It is straightforward to see that patterns do not form a group. However, we observe that the monoid of strong winding number ± 1 patterns has a natural inclusion into a group of homology cylinders modulo homology cobordism, introduced by J. Levine [Lev01]. We show that this group has a well-defined group action on knots in homology 3–spheres modulo a generalization of concordance, and that this action is compatible with the classical satellite construction as follows.

Theorem 5 ([DR16])**.** *Let $* =$ top or ex. Let $\mathcal{S}$ denote the monoid of strong winding number ± 1 patterns. There are groups $\widehat{\mathcal{C}_*}$ and $\widehat{\mathcal{S}_*}$, and functions $\mathcal{C}_* \hookrightarrow \widehat{\mathcal{C}_*}$ and $E : \mathcal{S} \to \widehat{\mathcal{S}_*}$, such that for any pattern $P \in \mathcal{S}$, the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{C}_* & \xrightarrow{P} & \mathcal{C}_* \\ \big\downarrow & & \big\downarrow \\ \widehat{\mathcal{C}_*} & \xrightarrow{E(P)} & \widehat{\mathcal{C}_*} \end{array}$$

Since $E(P)$ is an element of a group acting on $\widehat{\mathcal{C}_*}$, $E(P) : \widehat{\mathcal{C}_*} \to \widehat{\mathcal{C}_*}$ is a bijection.

As a consequence, we give a complete characterization of patterns inducing surjective satellite operators and show that there exists an infinite family of strong winding number ± 1 patterns, distinct from connected sum patterns, which induce bijections on $\mathcal{C}_*$. Moreover, for a large class of patterns P, we explicitly draw patterns $\overline{P}$ such that $\overline{P}(P(K))$ is concordant to K for each knot K.

We focus on winding number ± 1 satellites of a knot K with slice patterns, since these are difficult to distinguish from K in concordance. In this situation, the 0–surgery manifolds – obtained by performing 0–framed surgery on S^3 along the knot – are homology cobordant preserving the homology class of the positively oriented meridian, and as a result K and $P(K)$ have the same classical concordance invariants. Theorem 3 circumvents this difficulty via the slice–Bennequin inequality from contact geometry and modern smooth concordance invariants such as the $\tau-$ and s–invariants. Recently Yasui [Yas15] showed that there exist non-concordant knots with homeomorphic 0–surgeries, disproving a conjecture of Akbulut–Kirby from 1978 [Kir97, Problem 1.19]. These examples are satellite knots. In joint work with Feller and J. Park [FPR16], we use the Υ–invariant, a smooth invariant introduced by [OSS14], to extend this result as follows.

Theorem 6 ([FPR16])**.** *There is an infinite family of pairs of topologically slice knots $\{(K_i, J_i)\}_{i=0}^{\infty}$ such that the families $\{K_i\}_{i=0}^{\infty}$ and $\{J_i\}_{i=0}^{\infty}$ are each linearly independent in $\mathcal{C}$, each set $\{K_i, J_i\}$ is linearly independent in $\mathcal{C}$, and the 0–surgery manifolds for K_i and J_i are homeomorphic for each i.*

In a related vein, since a knot and its satellite under a slice winding number ± 1 pattern share several classical invariants, we may ask what we obtain when we identify such satellites. In [CR16], Cochran and I show that the equivalence relation on knots generated by concordance together with setting $K \sim P(K)$ for all K and all slice winding number one patterns P is the same as the equivalence relation generated by a generalization of concordance, called shake concordance, defined as follows. For any knot K, define a *shaking* of K to be a collection of 0–framed parallels of K, where $n+1$ of the parallels are co-oriented with K and the n remaining parallels are oppositely oriented, for some $n \geq 0$. The knots K_0 and K_1 are said to be *shake concordant* if there is a smoothly embedded planar surface A in $S^3 \times [0,1]$ cobounded by shakings of K_0 and K_1. Shake concordance is a relative version of *shake sliceness* of knots, defined by Akbulut in [Akb77]: a knot K is *shake slice* if a shaking of K bounds a smoothly embedded planar surface in B^4. There are no known examples of knots that are shake slice but not slice. However, we show the following in the relative case.

Theorem 7 ([CR16])**.** *There are infinitely many topologically slice knots which are pairwise shake concordant, but distinct in concordance.*

The above result follows from a complete characterization of shake concordance in terms of concordance and satellite operations.

Theorem 8 ([CR16]). *Two knots K and J are shake concordant if and only if there exist winding number one patterns P and Q with $P(U)$ and $Q(U)$ slice, such that $P(K)$ is concordant to $Q(J)$.*

Additionally, we obtain a characterization of shake slice knots in terms of the injectivity of winding number one satellite operator, which suggests a novel approach to addressing the long-open question of whether there exist shake slice knots that are not slice.

Corollary 9 ([CR16]). *There exists a shake slice knot that is not slice if and only if there exists a slice winding number one satellite operator $P : \mathcal{C} \to \mathcal{C}$ which fails to be weakly injective, i.e. there exists a non-slice knot K such that $P(K)$ is slice.*

2. Derivatives of slice knots

Any knot K is the boundary of an embedded surface in S^3 called its *Seifert surface*. The linking number of curves on the Seifert surface gives rise to the *Seifert form*, and if a knot has the Seifert form of a slice knot it is called *algebraically slice*. Clearly any slice knot is algebraically slice, but the converse is false [CG78, CG86]. However, given an algebraically slice knot K with a genus g Seifert surface F, we can find g–component links L on F such that if L were slice, we can perform surgery on F using the slice disks for L to obtain a slice disk for K. Such a link L is called a *derivative* of K on F. If F has genus one, there are exactly two derivatives, each of which is a knot. In 1982, Kauffman conjectured that a knot K with a genus one Seifert surface is slice if and only if it has a slice derivative [Kau87, p. 226]. Despite much supporting evidence, such as in [CHL10, COT03, Coo82, Gil93], Cochran–Davis [CD15] have recently shown that Kauffman's conjecture is false. In [Ray13], I showed that if a knot K which bounds a punctured Klein bottle F with 'zero framing' – a linking number condition trivially satisfied in the orientable case – there is a unique derivative curve J on F. In marked contrast with the orientable case, I showed that the natural non-orientable analogue of Kauffman's conjecture is true, as follows.

Theorem 10 ([Ray13]). *If a knot K bounds a Klein bottle with zero framing, the knot K is rationally slice if and only if the unique derivative J is rationally slice.*

Several concordance invariants obstruct rational sliceness; for example, the Levine–Tristram signature function and Ozsváth–Szabó's τ-invariant [OS03] are both zero for rationally slice knots. Therefore, unlike the punctured torus case, there are strong restrictions on the concordance class of derivatives on punctured Klein bottles bounded by slice knots.

Cochran–Davis's counterexamples to Kauffman's conjecture consist of genus one slice knots where the $\mathbb{Z}_2$–valued Arf invariant is non-zero for both derivatives, i.e. neither derivative is in $\mathcal{F}_0$. In [Par15], Park constructs genus one slice knots with derivatives J_1 and J_2 such that $\mathrm{Arf}(J_1) \neq 0$ and $J_2 \in \mathcal{F}_0 \setminus \mathcal{F}_{0.5}$. In ongoing joint work with Davis and J. Park [DPR16], we extend these results as follows.

Theorem 11 ([DPR16]). *For each $n > 0$, there exists a slice knot K with a genus one Seifert surface and derivatives J_1 and J_2 such that $\mathrm{Arf}(J_1) \neq 0$ and $J_2 \in \mathcal{F}_n \setminus \mathcal{F}_{n.5}$.*

Moreover, there exists a slice knot K with a genus one Seifert surface and derivatives J_1 and J_2 such that $\mathrm{Arf}(J_1) \neq 0$ and $J_2 \in \bigcap_{n=0}^{\infty} \mathcal{F}_n$.

Indeed, for the second statement in the theorem above, the derivative J_2 is topologically slice. This yields the following corollary.

Corollary 12 ([DPR16]). *There exists a slice knot K with a genus one Seifert surface F such that*

(1) K has a topological slice disk Δ_{top} such that $F \cup \Delta_{top}$ bounds a handlebody in B^4,
(2) K has no smooth slice disk Δ_{smooth} such that $F \cup \Delta_{smooth}$ bounds a handlebody in B^4.

3. Smooth concordance of topologically slice knots and links

Let $\mathcal{T}$ denote the group of smooth concordance classes of topologically slice knots. There are infinitely many topologically slice knots that are not smoothly slice [End95, Gom86, HK12, HLR12, Hom14]), i.e. $\mathcal{T}$ is highly non-trivial. However, while infinite rank subgroups of $\mathcal{T}$ have been known for many years, it was only recently that Hom showed the existence of a $\mathbb{Z}^\infty$ summand in $\mathcal{T}$ [Hom15], using her ε–invariant (this was soon reproved by Ozsváth–Stipsicz–Szabó in [OSS14] using their Υ–invariant). In joint work with Feller and J. Park [FPR16], we use the Υ–invariant to construct bases for infinite rank summands of $\mathcal{T}$ of knots with trivial Alexander polynomial.

Theorem 13 ([FPR16])**.** *Let K be a knot with $\tau(K) > 0$. Then $\{\mathrm{Wh}^+(K)_{2^i,1}\}_{i=0}^\infty$ is a basis for an infinite rank summand of $\mathcal{T}$, where Wh^+ denotes the positive Whitehead doubling operator.*

This should be compared to [KP16] where M. Kim–K. Park showed that $\{\mathrm{Wh}^+(RHT)_{n,1}\}_{n=2}^\infty$ is a basis for an infinite rank summand of $\mathcal{T}$, where RHT denotes the right-handed trefoil knot. There the crux of the proof was an explicit computation of part of the Υ–invariant of $\mathrm{Wh}^+(RHT)_{n,1}$, for $n \geq 2$. In contrast, our techniques are indirect; while we do not address the complete family of $(n, 1)$ cables of $\mathrm{Wh}^+(RHT)$, our result applies to a larger family of knots, since $\tau(RHT) > 0$.

For a link L, the *smooth (resp. topological) slice genus* is the minimal genus of a smooth (resp. locally flat) connected surface in B^4 with boundary L. This measures how far a link is from being slice since any slice link has zero slice genus. To complement the fact that the smooth and topological slice genera of knots can be far apart [Don83, CG88, Tan98, FM15], we show the following result for 2–component links in joint work with M. Kim and J. Park.

Theorem 14 ([KPR16])**.** *For any $i \geq 0$, there exists a 2–component topologically slice non-split boundary link ℓ_i with unknotted components such that the slice genus of ℓ_i is either i or $i + 1$.*

Recall that a link is slice if it is concordant to the trivial link. For 2–component links, in joint work with Davis, we study concordance to the Hopf link, and show that, similar to sliceness, there is indeed a sharp difference between the smooth and topological categories in this case.

Theorem 15 ([DR15])**.** *There exist infinitely many 2–component links with unknotted components which are topologically, but not smoothly, concordant to the Hopf link. Our examples are distinct from the previous examples given by Cha–Kim–Ruberman–Strle in* [CKRS12].

In recent years, the solvable filtration has been an invaluable framework for the study of knot and link concordance. However, it does not help us study smooth concordance classes of topologically slice knots and links, since these lie in the intersection of all the terms of the filtration [CHL09]. In this situation, we instead use the positive and negative filtrations, $\{\mathcal{P}_n\}_{n=0}^\infty$ and $\{\mathcal{N}_n\}_{n=0}^\infty$ respectively [CHH13], which do detect topologically slice knots and links [CHH13, CP14, CH15].

A motivating feature of the solvable filtration of $\mathcal{C}$ is its close relationships with several geometric filtrations of $\mathcal{C}$. For example, if a knot K bounds a *grope* of height $n + 2$, or a *Whitney tower* of height $n + 2$, in B^4 then $K \in \mathcal{F}_n$ [COT03]. In [Ray15a], I give similar geometric analogues for the positive and negative filtrations in terms of *Casson towers* [Cas86, Fre82]. Gropes, Casson towers, and Whitney towers are certain iterated 4–dimensional constructions. Casson towers, built out of layers of immersed disks, were used by Freedman in his proof of the 4–dimensional topological h–cobordism theorem and the 4–dimensional topological Poincaré Conjecture [Fre82]. Let $\{\mathcal{G}_n\}_{n=1}^\infty$ denote the grope filtration of $\mathcal{C}$. In [Ray15a], for $n \geq 1$, we say that a knot is in $\mathfrak{C}_n$ if it bounds a Casson tower of height n in B^4, and in $\mathfrak{C}_n^+$ (resp. $\mathfrak{C}_n^-$) if it bounds a Casson tower of height n in B^4 where the self-intersections in the base level disk are all positive (resp. negative). We similarly define the filtrations $\{\mathfrak{C}_{2,k}\}_{k=0}^\infty$, $\{\mathfrak{C}_{2,k}^+\}_{k=0}^\infty$, and $\{\mathfrak{C}_{2,k}^-\}_{k=0}^\infty$. In [Ray15a] I established the following relationships between the various filtrations of $\mathcal{C}$. (The second inclusion in part (i) was proved in [COT03]; we include it here for completeness.)

Theorem 16 ([Ray15a]). *For any $n \geq 0$,*

(i) $\mathfrak{C}_{n+2} \subseteq \mathcal{G}_{n+2} \subseteq \mathcal{F}_n$, *and* $\mathfrak{C}_{2,n} \subseteq \mathcal{F}_n$,
(ii) $\mathfrak{C}^+_{n+2} \subseteq \mathfrak{C}^+_{2,n} \subseteq \mathcal{P}_n$, *and* $\mathfrak{C}^-_{n+2} \subseteq \mathfrak{C}^-_{2,n} \subseteq \mathcal{N}_n$.

Mirroring the fact that the positive and negative filtrations non-trivially filter smooth concordance classes of topologically slice knots and links [CHH13, CP14, CH15], it is expected that the filtrations $\{\mathfrak{C}^{\pm}_{2,n}\}_{n=0}^{\infty}$ will as well.

One of the ingredients of the above theorem is the following result, which was a key element in Cha–Powell's proof [CP16] that any Casson tower of height four contains an embedded topological slice disk for its attaching circle, and that any link whose components bound a particular type of Casson tower of height three in B^4 is topologically slice.

Theorem 17 ([Ray15a]). *Any Casson tower of height n contains a grope of height n with the same attaching circle.*

We also obtain the following result about Casson towers and the intersection of the terms in the solvable filtration.

Theorem 18 ([Ray15a]). *If a knot K bounds a Casson tower of height three, then $K \in \bigcap_{n=0}^{\infty} \mathcal{F}_n$.*

The only presently known elements of $\bigcap \mathcal{F}_n$ are topologically slice and it is an open question whether all knots in $\bigcap \mathcal{F}_n$ are topologically slice. Thus, by the above result, either every knot that bounds a Casson tower of height three in B^4 is topologically slice or there exist knots in $\bigcap \mathcal{F}_n$ which are not topologically slice. (Note that by [CP16] any knot that bounds a Casson tower of height four is topologically slice.)

4. Dehn's lemma in dimension four

The classical 3–dimensional Dehn's lemma [Pap57] says that if an embedded circle in the boundary of a 3–manifold is null-homotopic in the interior then it is in fact the boundary of an embedded disk. Analogously, for 4–manifolds, we may ask about embedded circles in the boundary, or about embedded codimension one submanifolds of the boundary. The former is essentially a question about knot concordance. In joint work with Ruberman, we consider the latter situation in [RR16]. Here, unlike the 3–dimensional case, the distinction between the smooth and topological categories for 4–manifolds introduces an added element of complexity. We establish the following result in the case of spheres.

Theorem 19 ([RR16]). *Dehn's lemma does not hold for spheres in the boundary of 4–manifolds.*

However, any sphere in a homology sphere boundary of a 4–manifold with abelian fundamental group does extend to an embedded topological ball in the interior.

Additionally, there exist spheres in the boundary of simply connected 4–manifolds that extend to an embedded topological ball in the interior, but no smoothly embedded ball.

We also study tori in the boundary of 4–manifolds. Note that a sphere is null-homotopic exactly when it extends to a map of a ball. Thus, for tori, we consider those which extend to maps of the solid torus. Once again, we show that the analogue of Dehn's lemma does not hold.

Theorem 20 ([RR16]). *There exist incompressible tori in the boundary of contractible 4-manifolds such that they extend to a map of a solid torus in the interior, but do not bound embedded topological solid tori in the interior.*

However, there exist such tori which bound embedded topological solid tori in the interior, but no smooth solid torus.

5. Ongoing and future work

5.1. **Handlebody-n–solvable knots.** (Joint work with Davis and J. Park) A convenient method to detect a slice knot consists of finding a slice link as a derivative on a Seifert surface; as we saw in Section 2, the strategy is to perform surgery on the Seifert surface using the slice disks for the link to obtain a slice disk for the original knot. The converse to this, that any slice knot is of this form, is the famous Kauffman Conjecture, which was disproved recently by Cochran–Davis [CD15]. The above process in fact yields more than just a slice disk for the knot – it provides a handlebody in B^4 bounded by the closed surface obtained by capping off the Seifert surface with the slice disk resulting from surgery.

Let L be a derivative link for a knot K. There is a filtered version to the above construction: for any n, if $L \in \mathcal{F}_n$, then $K \in \mathcal{F}_{n+1}$ [COT03]. The converse to this statement – the filtered Kauffman Conjecture – was also disproved by Cochran–Davis in [CD15]. A natural question then is to ask what more must be required of K in order to imply the existence of a derivative link one step lower in the solvable filtration. As before, if a knot K has a derivative link in $\mathcal{F}_n$, then a capped off Seifert surface for K bounds a handlebody in a certain 4–manifold. By definition, a knot $K \in \mathcal{F}_n$ exactly when it bounds a disk in a 4–manifold of a particular form called an *n–solution*. We propose a new filtration on knots based on when a capped off Seifert surface bounds a handlebody in an n–solution, possibly with certain other requirements. Such a knot might be called *handlebody-n–solvable*.

Problem 21. Construct a new filtration of the knot concordance group, with respect to which the filtered Kauffman Conjecture is true.

Recall that any ribbon knot R bounds a Seifert surface F such that if Δ is a ribbon disk for R, then $F \cup \Delta$ bounds a handlebody in B^4. Therefore, as the solvable filtration determines how close a knot is to being slice, it is conceivable that the proposed filtration will detect how close a knot is to being ribbon.

5.2. **Satellite operators as group actions.** (Joint work with Davis) The set of patterns has a natural binary operation $\star$ which gives it the structure of a monoid such that the satellite construction is a monoid action, i.e. $(P \star Q)(K) = P(Q(K))$ for any patterns P and Q and knot K. Davis and I observed in [DR16] that the monoid of strong winding number ± 1 patterns has a natural inclusion into the group of *generalized patterns*, denoted $\widehat{\mathcal{S}}_{\text{top}}$ or $\widehat{\mathcal{S}}_{\text{ex}}$ depending on the category. As discussed in Section 1, the classical satellite operation is the restriction of a group action by generalized patterns on concordance classes of knots in homology spheres. This repackaging yields several interesting results, and we believe that further insights can be gained from this perspective.

The Mazur pattern can be used to show that that the group $\widehat{\mathcal{S}}_{\text{ex}}$ is not abelian; however, it is not known whether $\widehat{\mathcal{S}}_{\text{top}}$ is abelian.

Problem 22. Study the group-theoretic properties of the groups of generalized patterns.

For example, is $\widehat{\mathcal{S}}_{\text{top}}$ abelian? Are there any higher-order commutators? Do connected sum operators form a normal subgroup? Is there any non-trivial torsion i.e. not given by connected sum with finite order knots? What is the center?

It is also interesting to study the group action itself as a generalization of the classical satellite construction. It is straightforward to see that the group action is transitive by using connected sum operators.

Problem 23. Study the group action of generalized patterns on concordance classes of knots in homology 3–spheres.

For example, is the action faithful? free? n–transitive? Etc.

We currently have some partial results in this direction. In ongoing work, Davis and I use the notion of n–solve equivalence of bordered 3–manifolds given by Jae Choon Cha in [Cha14] to give a collection of equivalence relations on generalized patterns. We show that n–solve equivalence classes of generalized patterns have a well-defined action on n–solve equivalence classes of knots, and that this action is not faithful, i.e. for each n, there exist generalized patterns P and Q such that P and Q are not n–solve equivalent but $P(K)$ is concordant to $Q(K)$ for any knot K in a homology sphere. This indicates that the generalized satellite operation is unlikely to be faithful.

5.3. **Reversibility of knots in concordance.** (Joint work with M. Kim and J. Park) Given a knot K, its *reverse*, denoted rK, is obtained by reversing the orientation of the knot. A knot is said to be *reversible* if it is isotopic to its reverse; the existence of non-reversible knots was first shown in 1963 by Trotter in [Tro63], and the smallest such knot is 8_{17}. The question of whether a knot is concordant to its reverse is rather delicate since e.g. recently developed powerful concordance invariants fail to detect orientation. Nonetheless, examples of knots that are non-reversible in concordance were given by [Liv83, Nai96, KL99, Tam99, HKL10, CKL15]. In general, these results show that certain specific knots are not concordant to their reverses by performing a computation of either a Casson–Gordon invariant or a twisted Alexander polynomial, as opposed to constructing infinite families of such knots.

Problem 24. Construct an infinite family of knots that are non-reversible in concordance.

Since torus knots are reversible, the reverse of the (p, q)–cable of a knot K is the (p, q)–cable of the reverse of K, i.e. $r(K_{p,q}) = (rK)_{p,q}$. Moreover, for a pattern P with non-zero winding number, the knots $P(K)$ and $P(J)$ are rationally concordant if and only if K and J are rationally concordant [CDR14]. Knots under rational concordance form an abelian group denoted $\mathcal{C}_\mathbb{Q}$. Consequently, for a given knot K, if $K_{p,q}$ is reversible in concordance, i.e. $K_{p,q} = r(K_{p,q}) = (rK)_{p,q}$ in $\mathcal{C}$ and therefore in $C_\mathbb{Q}$, then K is rationally concordant to rK. Therefore, if any of the above known examples of knots which are non-reversible in concordance also fail to be reversible in rational concordance, then their cables would form an infinite family of knots that are non-reversible in concordance, as desired. Thus, one approach to our problem would be to determine whether the obstructions used in the papers mentioned above also obstruct rational concordance.

Question 25. Do Casson–Gordon invariants or twisted Alexander polynomials obstruct rational concordance?

5.4. **Distance between satellite operators.** (Joint work with M. Kim and J. Park) Reversing the orientation of a knot is the result of applying a winding number -1 satellite operator, called the *reverse* operator. In [CH14], Cochran–Harvey studied the set of knot concordance classes as a metric space: for knots K and J, define the metric $d_s(K, J)$ to be the minimal genus of an oriented surface in $S^3 \times [0, 1]$ cobounded by K and J. Under this metric, the identity and reverse operators clearly give bijective isometries on $\mathcal{C}$. Cochran–Harvey show that no winding number m satellite operator is within a bounded distance of any winding number n satellite operator if $m \neq \pm n$. It is then interesting to ask whether the $\pm$ here can be replaced by just $+$. For $m = 1$, the question is whether the identity operator is within a bounded distance of the reverse operator, even though each is a bijective isometry of $\mathcal{C}$. We expect that the family of cable knots developed in Problem 24 should provide the examples needed for the following.

Problem 26. Show that the reverse operator is not within a bounded distance of the identity operator.

An answer to the above would imply that no winding number 1 satellite operator is within a bounded distance of a winding number -1 satellite operator. We generalize this as follows.

Problem 27. Show that no winding number m satellite operator is within a bounded distance of a winding number $-m$ satellite operator.

5.5. **Satellite operations and linear dependence of knot concordance classes.** (Joint work with M. Kim, J. Park, and K. Park) Recall that satellite operators are not group homomorphisms in general. As a result, a priori we do not expect them to respect linear independence of knot concordance classes. In joint work with Feller and J. Park, we studied when satellite operators preserve linear independence, as discussed in Section 1. In a complementary vein, in joint work with M. Kim, J. Park and K. Park, we study whether it is possible for a satellite operator to map a linearly dependent set to a linearly independent set.

Question 28. For any integer w, is there an unknotted winding number w pattern P and a knot K such that $\{P(nK) \mid n \geq 1\}$ is linearly independent?

We have the following partial result in this direction.

Theorem 29 ([KPPR16])**.** *For any integer w, there exists an unknotted pattern P with winding number w and a topologically slice knot K such that $\{P(K), P(2K)\}$ is linearly independent.*

For $w = 0$, the pattern above is the Whitehead doubling pattern. We are particularly interested in this pattern, and hope to show that the $\{\mathrm{Wh}^+(nD)\}_{n=1}^{\infty}$ is a linearly independent set, where D denotes the positive Whitehead double of the right-handed trefoil.

5.6. **Concordance of knots in $S^1 \times S^2$.** (Joint work with Davis, Nagel, and J. Park) Consider a knot in $S^1 \times S^2$, i.e. a smooth embedding of an oriented S^1 in $S^1 \times S^2$. We say that two knots K and J in $S^1 \times S^2$ are *concordant* (resp. *topologically concordant*) if they cobound a smooth (resp. locally flat) annulus in $S^1 \times S^2 \times I$. The knot $S^1 \times \{p\} \subset S^1 \times S^2$ is the *Hopf knot H*. Note that once we choose a generator for $\pi_1(S^1 \times S^2)$, any knot in $S^1 \times S^2$ has a well-defined *winding number* corresponding to its homotopy class, and winding number is preserved under concordance.

Problem 30. Compare smooth and topological concordance of knots in $S^1 \times S^2$.

We learned of this problem through the recent work of Friedl–Nagel–Orson–Powell [FNOP16], where they show that any winding number one knot in $S^1 \times S^2$ is topologically concordant to H. We strengthen their result as follows.

Theorem 31 ([DNPR16])**.** *Any winding number one knot in $S^1 \times S^2$ is smoothly concordant to the Hopf knot.*

Call a knot in $S^1 \times S^2$ *slice* (resp. *topologically slice*) if it bounds a smooth (resp. locally flat) embedded disk in $D^2 \times S^2$. Clearly, the Hopf knot H is slice. In this setting, there are infinitely many non-concordant slice knots, since winding number is an obstruction. Therefore, we consider the $(p, 1)$ cable of H to be the trivial knot with winding number p, for any $p \in \mathbb{Z}$.

Problem 32. For any winding number w, find examples of winding number w knots that are slice but not concordant to the trivial winding number w knot. Find knots that are topologically concordant to the trivial winding number w knot, but not smoothly slice.

Note that patterns, i.e. knots in solid tori $S^1 \times D^2$ are a natural subset of knots in $S^1 \times S^2$. Indeed, using our result mentioned above, we see that there are pattern knots that are non-concordant in $S^1 \times D^2 \times I$ (e.g. the Mazur pattern and the trivial pattern), which are nonetheless smoothly concordant when considered as knots in $S^1 \times S^2$. Therefore, we consider this project to be an approach towards better understanding concordance of knots in solid tori, and hence, satellite operators on concordance classes of knots in S^3.

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