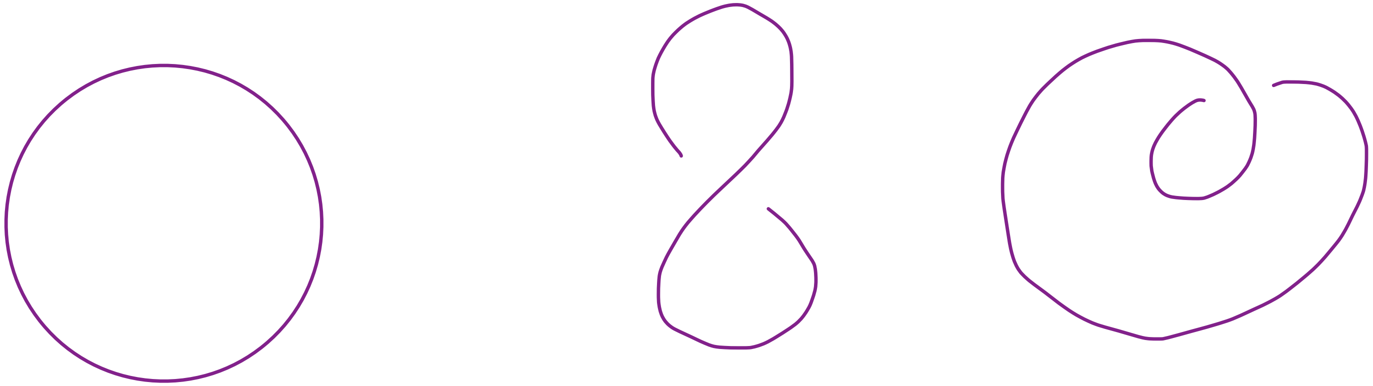
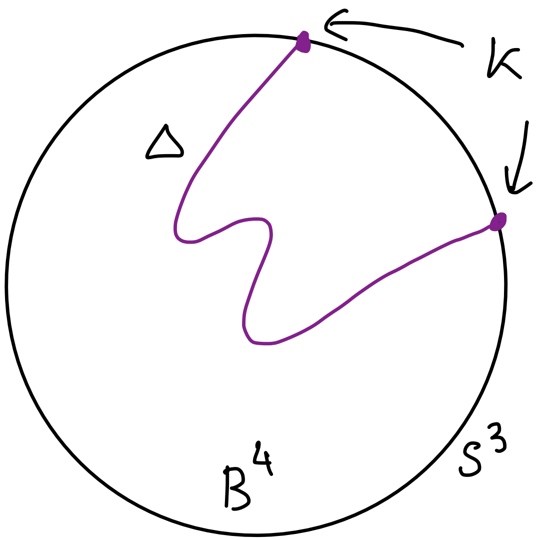
Building bridges seminar
February 17, 2021

Filtrations of the knot concordance group

Slice knots

A knot $K \subseteq S^3$ is trivial iff it bounds an embedded disc in S^3



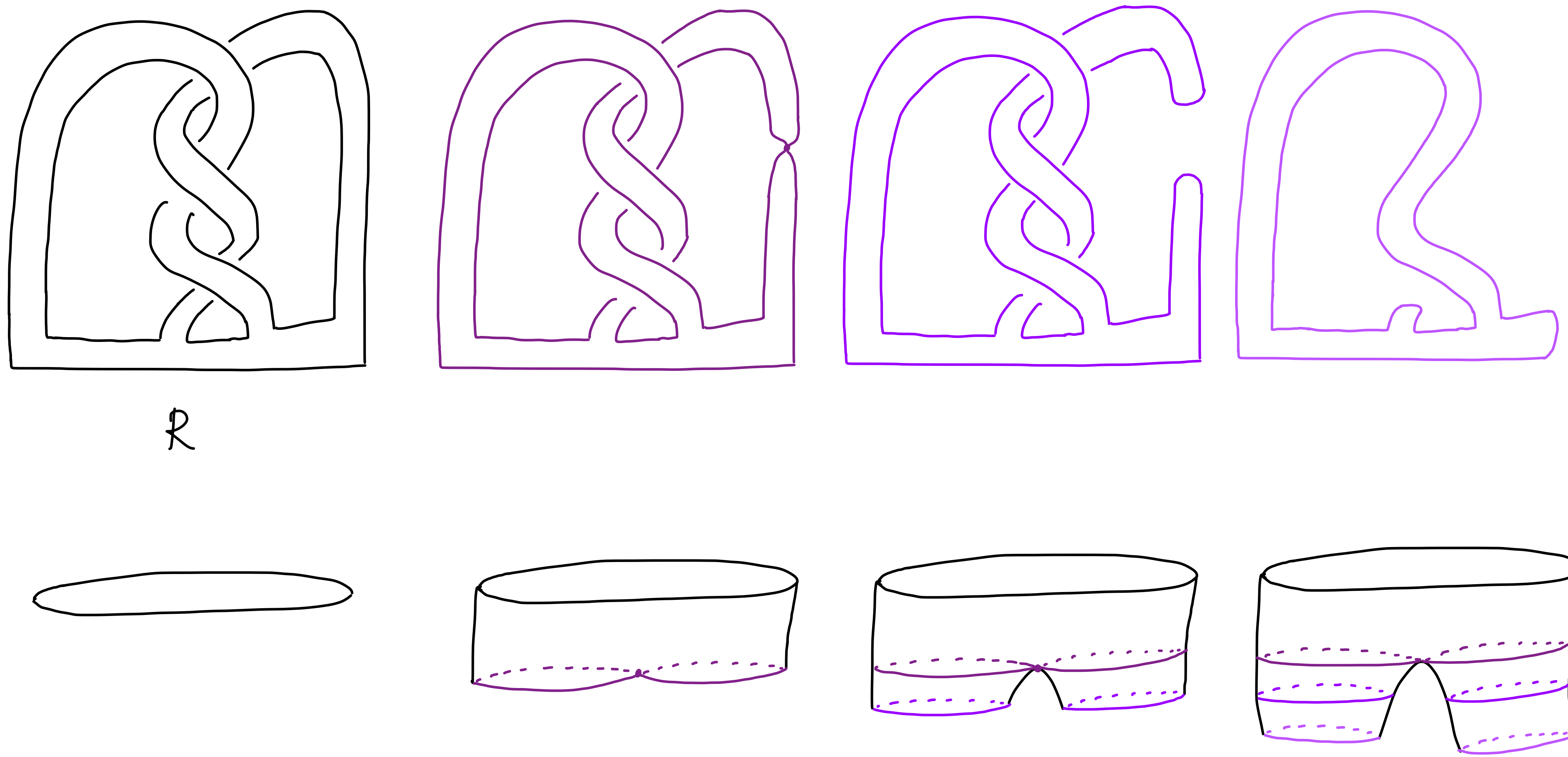


Consider $K \subseteq S^3$ bounding an embedded disc $\Delta \subseteq B^4$

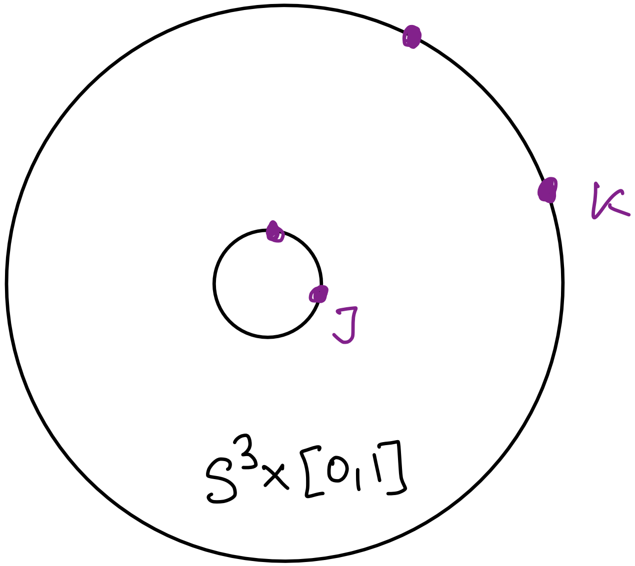
- if Δ is smooth, K is smoothly slice
- if Δ is flat, K is topologically slice

Trivial $\overset{\Longrightarrow}{\not\Longleftarrow}$ slice

Examples of slice knots



Concordance of knots



$\mathcal{C} := \left\{ \begin{array}{l} \text{conc. classes} \\ \text{of knots} \end{array} \right\}$ is a group under $\#$

Obstructions to sliceness

Not all knots are slice

e.g. slice $\Longrightarrow$

Goal: organise these systematically

Solvable filtration of $\mathcal{C}$ (Cochran-Orr-Teichner 2003)

$$\left\{\begin{matrix}\text{slice}\\ \text{knots}\end{matrix}\right\} \subseteq \bigcap \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}_{n.5} \subseteq \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}_{0.5} \subseteq \mathcal{F}_0 \subseteq \mathcal{C}$$

Some properties

$$\left\{\begin{matrix}\text{slice}\\ \text{knots}\end{matrix}\right\} \subseteq \bigcap^{\infty} \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}_{n.5} \subseteq \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}_{0.5} \subseteq \mathcal{F}_0 \subseteq \mathcal{C}$$

$$\mathcal{F}_0 = \{ K \mid \qquad \}$$

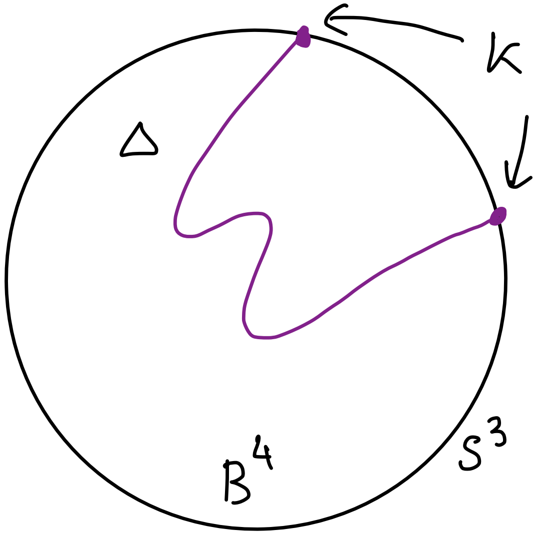
$$\mathcal{F}_{0.5} = \{ K \mid \qquad \}$$

$$\mathcal{F}_{1.5} \subseteq \{ K \mid \qquad \}$$

Jiang
Livingston
Cochran-Orr-Teichner
Cochran-Teichner
Cochran-Harvey-Leidy
see also Cha, Davis-Park-R.

Definition of $\{\mathcal{F}_n\}$

Motivation: wish to approximate sliceness

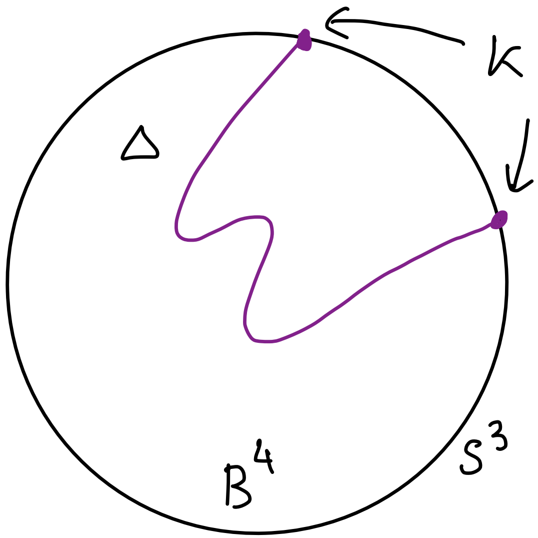


K is slice if it bounds a disc inside B^4

K TOP slice $\iff$ K bounds a disc in TOP W^4 with $\partial W^4 = S^3$

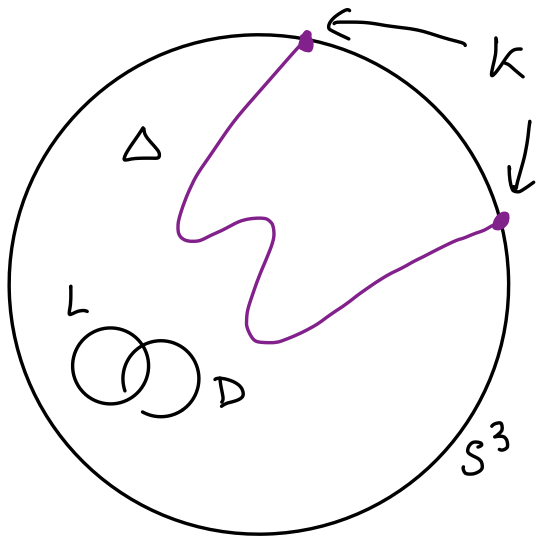
s.t. $\pi_1 W = 1$

$H_2 W = 0$



K TOP slice

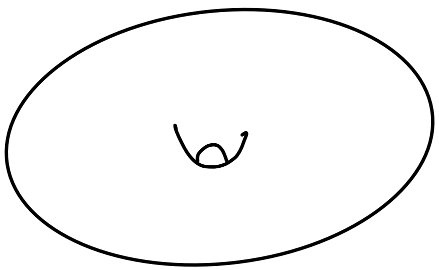
$$\xRightarrow{\# S^2\times S^2}$$



K bounds a disc Δ in TOP W^4 s.t. $\pi_1 W = 1$

$H_2(W)$ gen by emb spheres $\{L, D\}$ w. trivial normal bundle and $L \pitchfork D = pt$

$L, D \cap \Delta = \emptyset$



$S^2 \times S^2$

Definition: K is *n-solvable*, denoted $K \in \mathcal{F}_n$, if it bounds a disc Δ in a TOP W^4 s.t.

1. $H_1(W) = 0$

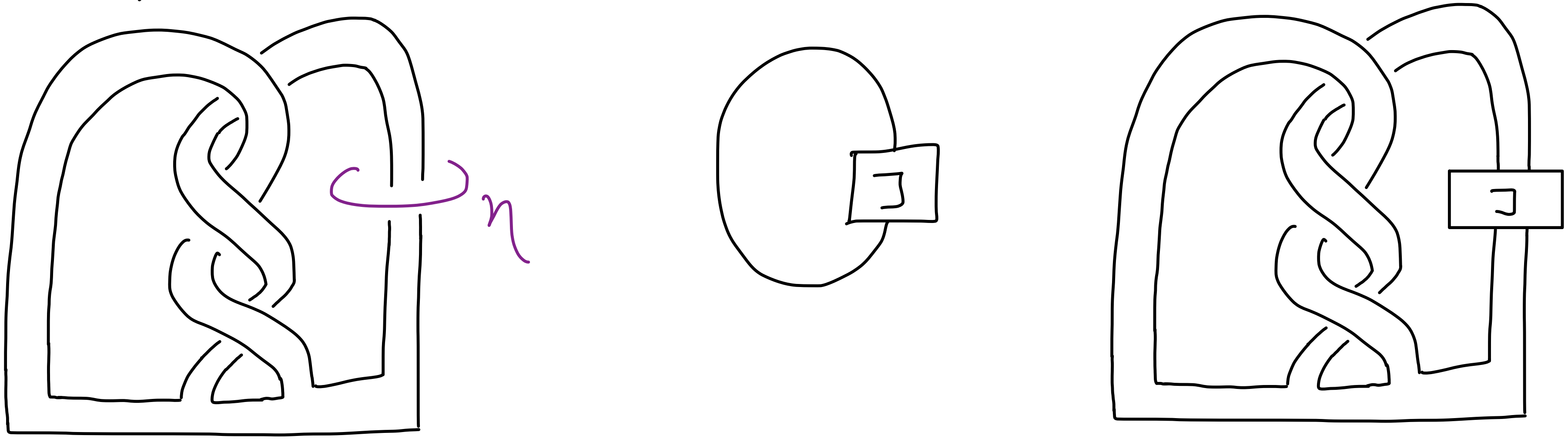
2. $H_2(W)$ gen by embedded *surfaces* $\{L_i, D_i\}$, $L_i, D_i \subseteq W \setminus \Delta$ w. trivial normal bundle s.t. $L_i \pitchfork D_j = \delta_{ij}$,
$$L_i \cap L_j = 0 = D_i \cap D_j$$

3. $\pi_1(L_i) \subseteq \pi_1(W \setminus \Delta)^{(n)}$
$\pi_1(D_i) \subseteq \pi_1(W \setminus \Delta)^{(n)}$

if in addition $\pi_1(L_i) \subseteq \pi_1(W \setminus \Delta)^{(n+1)}$, then K is *n.5 solvable* denoted $K \in \mathcal{F}_{n.5}$

Examples

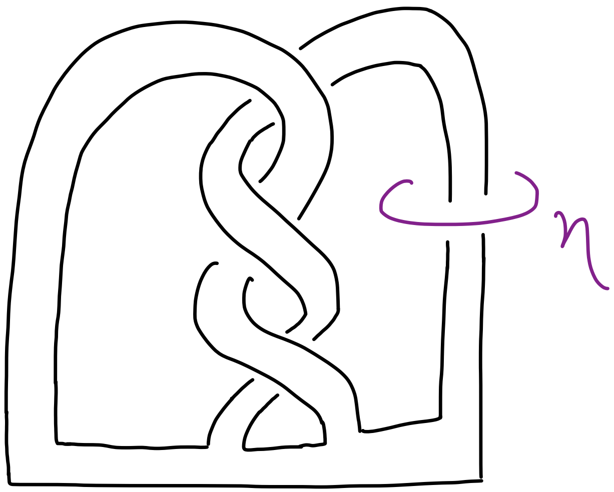
Infection/satellite operation:



R $\qquad\qquad\qquad\qquad$ $R_\eta(J)$

if $R \in \mathcal{F}_n^U$ & $\eta \in \pi_1(S^3 \setminus R)^{(k)}$ then $R_\eta(\mathcal{F}^U_{n-k}) \subseteq \mathcal{F}^U_n$

Proof: if $R \in \mathcal{F}_n$, $\eta \in \pi_1(S^3 \setminus R)^{(k)}$ then $R_\eta(\mathcal{F}_{n-k}) \subseteq \mathcal{F}_n$



R

Let $J \in \mathcal{F}_{n-k}$ $\quad R \subseteq S^3 \setminus \eta \times D^2$

$R = \partial \Delta_R$ in some W_R n-solution

$J = \partial \Delta_J$ in some W_J $(n-k)$-solution

$$W_{R_\eta(J)} = W_R \cup W_J \setminus (\Delta_J \times D^2)$$

$\eta \times D^2 = \Delta_J \times S^1$

$\eta \mapsto \mu_J$

Obstructions

M^3 closed, oriented, Γ discrete group

define $\rho(M, \varphi : \pi_1 M \longrightarrow \Gamma) := \sigma^{(2)}_{\Gamma}(W, \psi) - \sigma(W)$

where $M = \partial W^4$. W compact, oriented

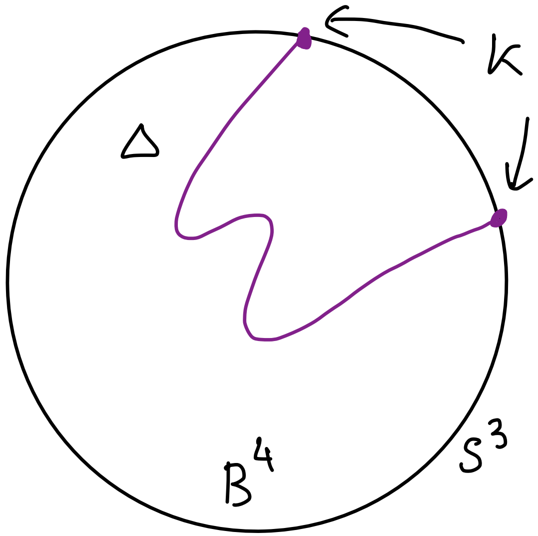
$$\begin{tikzcd}
\pi_1 M \arrow[r, "\varphi"] \arrow[d, "i_*"'] & \Gamma \\
\pi_1 W \arrow[ur, "\psi"'] &
\end{tikzcd}$$

[Cochran-Orr-Teichner] $K \in \mathcal{F}_{n.5}$ and Γ is PTFA with $\Gamma^{(n+1)} = 0$

then $\rho(S^3_0(K), \varphi : \pi_1(S^3_0(K)) \longrightarrow \Gamma) = 0$

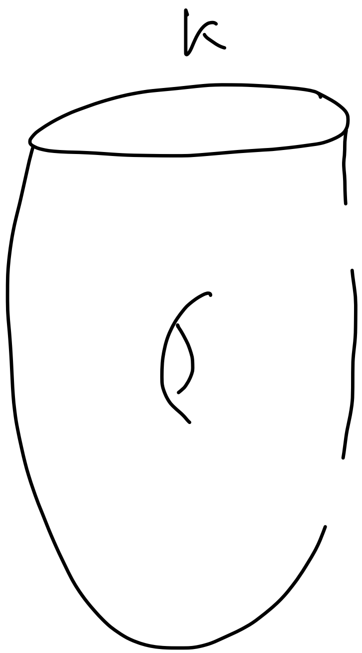
Other approximations?

Motivation: wish to approximate sliceness



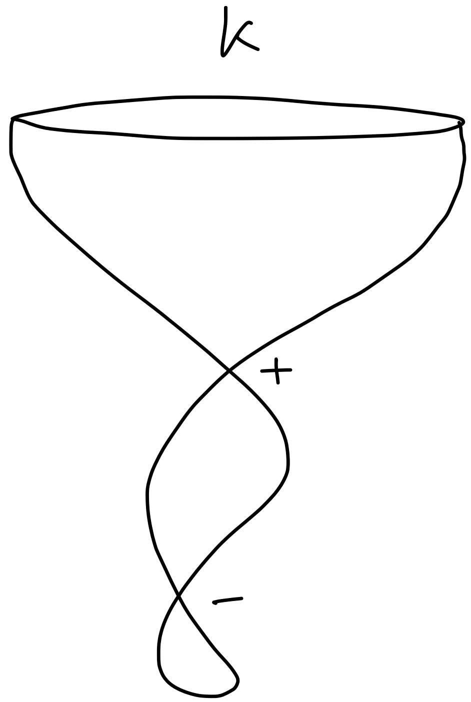
K is slice if it bounds a disc inside B^4

Gropes



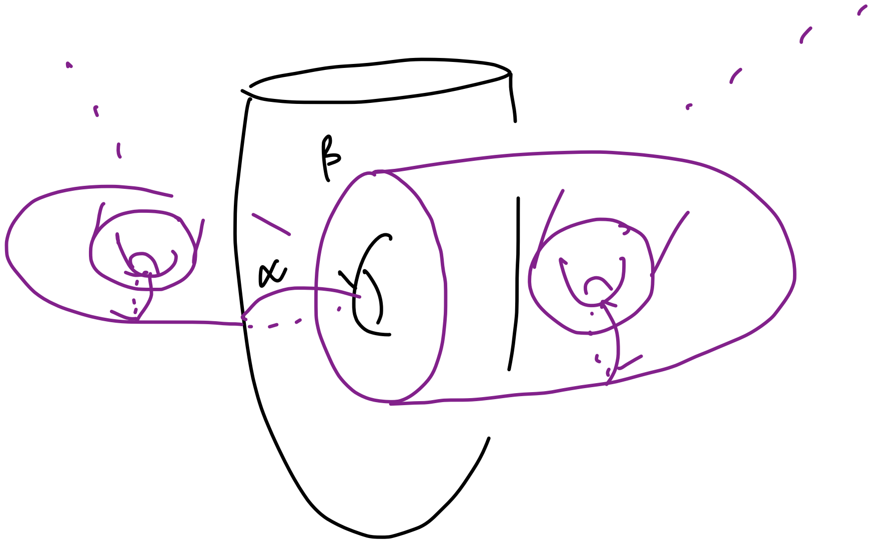
$K \in \mathcal{G}_n$ if it bounds a height n grope in B^4.

Whitney towers



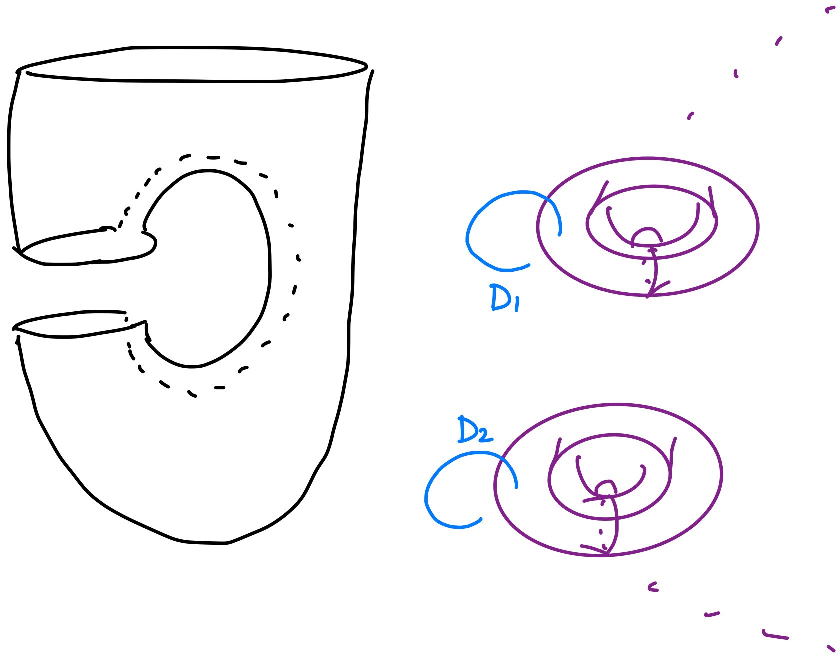
$K \in W_n$ if it bounds a height n Whitney tower in B^4

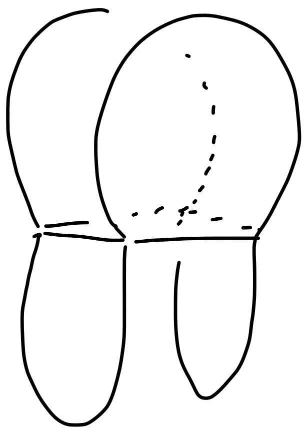
$\mathcal{G}_{n+2} \subseteq \mathcal{F}_n \ \forall n$ [Cochran-Orr-Teichner]



$\longrightarrow$

$\setminus S^1 \times D^3 \cup D^2 \times S^2$ along α, β.



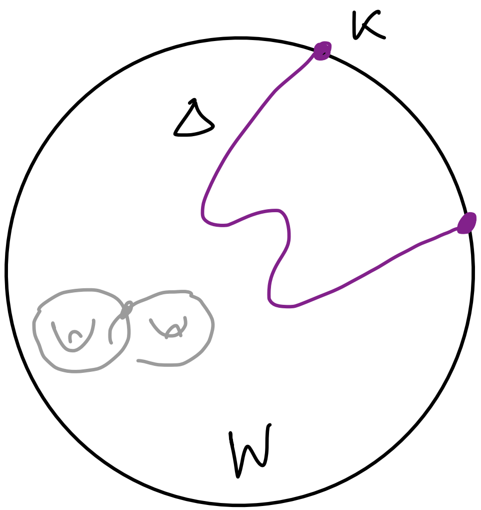


Similarly $\mathcal{W}_{n+2} \subseteq \mathcal{F}_n \ \forall n$

Smooth vs topological concordance

$$\mathcal{F}_n^{sm} / \mathcal{F}_{n.5}^{sm} \cong \mathcal{F}_n^{TOP} / \mathcal{F}_{n.5}^{TOP} \quad \forall n$$

$$K \in \mathcal{F}_n^{sm} \iff K \in \mathcal{F}_n^{TOP}$$



$K = \partial \Delta$ with $\Delta \subseteq W^4$ TOP

Check: $ks(W) = 0$

Positive / negative / bipolar filtrations

$K \in \mathcal{P}_n$ if it bounds a disc Δ in a smooth W^4 s.t.

1. $\pi_1(W) = 0$

2. $H_2(W)$ gen by embedded surfaces $\{S_i\}$, $S_i \subseteq W \setminus \Delta$, $S_i \cap S_j = \emptyset \ \forall i \neq j$
   intersection form is positive definite

3. $\pi_1(L_i) \subseteq \pi_1(W \setminus \Delta)^{(n)}$
   $\pi_1(D_i) \subseteq \pi_1(W \setminus \Delta)^{(n)}$

$K \in \mathcal{N}_n$ if

$K \in \mathcal{B}_n$ if

Smooth vs topological concordance

Let $\mathcal{T} := \left\{ \begin{array}{l} \text{smooth concordance classes} \\ \text{of topologically slice knots} \end{array} \right\}$

Define $\mathcal{T}_n := \mathcal{B}_n \cap \mathcal{T} \quad \forall n$

$\left.\begin{array}{l} \text{Cochran-Harvey-Horn} \\ \text{Cochran-Horn} \\ \text{Cha-Kim} \end{array}\right\}$

(see also Cha-Powell)

Miscellaneous results and open questions.

Generalisations

- Links?
  - string link concordance group
  - define $\mathcal{F}_n$, $\mathcal{G}_n$, W_n, P_n, N_n, $\mathcal{B}_n$, T_n similarly.

- Double concordance group
  - analogues for $\mathcal{F}_n$, $\mathcal{G}_n$, W_n. [T. Kim, Cha-Kim]
  - smooth vs TOP?

Nontriviality

- $\exists\, \mathbb{Z}^\infty \oplus \mathbb{Z}/2^\infty \subseteq \mathcal{G}_n/\mathcal{G}_{n+1} \quad \forall n$ [Horn, Jang]

- $\mathcal{F}^m_n/\mathcal{G}^m_{n+2} \neq 0$ for $m \geq 2^{n+2}$ [Otto]

  What about for knots?

- $\mathcal{F}^m_{n.5}/\mathcal{F}^m_{n+1} \neq 0$ for $m \geq 3 \cdot 2^{n+1}$ [Otto]

What about for knots?

Every genus 1 knot in $\mathcal{F}_{0.5}$ is in $\mathcal{F}_1$ [Davis-Martin-Otto-Park]

- $\mathcal{G}_n \subseteq W_n \ \forall n$ [Schneiderman]

  are they equal?

- Geometric analogue for P_n, N_n, B_n?
  - in terms of Casson towers [R.]

- Does there exist $\mathbb{Z}/2^\infty \subseteq \mathcal{T}_n / \mathcal{T}_{n+1} \ \forall n$?
  - $\mathbb{Z}/2^\infty \subseteq \mathcal{T}_0/\mathcal{T}_1$ [Chen]

- $\{\text{TOP slice}\} \subseteq \bigcap \mathcal{T}_n$

  are they equal?

Characterisation

- $\mathcal{F}_0^{\mathcal{V}} = \{K \mid \mathrm{Arf}(K) = 0\}$,
- $\mathcal{F}_{0.5}^{\mathcal{V}} = \{K \mid \text{alg slice}\}$

[Cochran-Orr-Teichner]

- $\mathcal{F}_0^{\mathcal{V}m}$ characterised via Milnor invts [Martin]

- $\mathcal{F}_{0.5}^{\mathcal{V}m}$? $\mathcal{P}_0, \mathcal{N}_0, \mathcal{B}_0$?
  - $\tilde{\mathcal{P}}_0$ in terms of gen. crossing changes [Cochran-Tweedy]

Interaction with other properties

- $\exists ?\ K \in \mathcal{T}_n$ with large g_4?

  $n=2$ [Cha-Miller-Powell]

  smooth version?

- $\exists ?\ K \in \mathcal{T}_n$, $K \not\sim K^r$?

  $\vdots$

Proxy for sliceness/concordance

- Is every knot in a $\mathbb{Z}HS^3$ TOP conc. to a knot in S^3?

  Yes, "up to solvable filtration" [Davis]

Questions?