Joint Math Meetings 2021

Jan 8, 2021

Isotopy and equivalence of knots in 3-manifolds

w. Aceto, Bregman, Davis, Park

Setup -

Let Y be a closed, oriented 3-manifold

A **knot** is a (tame) embedding $S^1 \hookrightarrow Y$

oriented

Setup-

Let Y be a closed, oriented 3-manifold

A **Knot** is a (tame) embedding $S^1 \hookrightarrow Y$

oriented

Knots K and J are

- **equivalent** if $\exists f : Y \to Y$ o.p. homeo s.t. $f \circ K = J$

Setup-

Let Y be a closed, oriented 3-manifold

A **Knot** is a (tame) embedding $S^1 \hookrightarrow Y$

oriented

Knots K and J are

- **equivalent** if $\exists f : Y \to Y$ o.p. homeo s.t. $f \circ K = J$

- **isotopic** if $\exists F_t : S^1 \hookrightarrow Y$ (tame) 1-par family of embs s.t. $F_0 = K$ and $F_1 = J$

Setup.

Let Y be a closed, oriented 3-manifold

A **Knot** is a (tame) embedding $S^1 \hookrightarrow Y$

oriented

Knots K and J are

- **equivalent** if $\exists f : Y \to Y$ o.p. homeo s.t. $f \circ K = J$

- **isotopic** if $\exists F_t : S^1 \hookrightarrow Y$ (tame) 1-par family of embs s.t. $F_0 = K$ and $F_1 = J$

- **ambient isotopic** if $\exists G_t : Y \to Y$ 1-par family of homeo s.t. $G_0 = id_Y$ and $G_1 \circ K = J$

For knots $K, J : S^1 \hookrightarrow Y$,

Isotopy	Ambient isotopy	Equivalence
$\exists F_t : S^1 \hookrightarrow Y$ s.t. $F_0 = K$, $F_1 = J$	$\exists G_t : Y \to Y$ s.t. $G_0 = id_Y$, $G_1 \circ K = J$	$\exists f : Y \xrightarrow{\cong} Y$ o.p s.t. $f \circ K = J$

For knots $K, J : S^1 \hookrightarrow Y$,

[Edwards-Kirby]

iso. extension theorem

Isotopy $\Longrightarrow$ **Ambient isotopy**

Isotopy $\Longleftarrow$ **Ambient isotopy**

$f_t := G_t \circ K$

Equivalence

Isotopy: $\exists f_t : S^1 \hookrightarrow Y$ s.t. $f_0 = K$, $f_1 = J$

Ambient isotopy: $\exists G_t : Y \to Y$ s.t. $G_0 = id_Y$, $G_1 \circ K = J$

Equivalence: $\exists f : Y \xrightarrow{\cong} Y$ o.p s.t. $f \circ K = J$

For knots $K, J : S^1 \hookrightarrow Y$,

$$\begin{tikzcd}[column sep=huge] \boxed{\text{Isotopy}} \arrow[r, Rightarrow, shift left=2, "\substack{\text{[Edwards-Kirby]} \\ \text{iso. extension} \\ \text{theorem}}"] & \boxed{\text{Ambient isotopy}} \arrow[l, Rightarrow, shift left=2, "f_t := G_t \circ K"] \arrow[r, Rightarrow, "f := G_1"] & \boxed{\text{Equivalence}} \end{tikzcd}$$

Isotopy: $\exists f_t : S^1 \hookrightarrow Y$ s.t. $f_0 = K$, $f_1 = J$

Ambient isotopy: $\exists G_t : Y \to Y$ s.t. $G_0 = id_Y$, $G_1 \circ K = J$

Equivalence: $\exists f : Y \xrightarrow{\cong} Y$ o.p s.t. $f \circ K = J$

For knots $K, J : S^1 \hookrightarrow Y$,

$$\boxed{\text{Isotopy}} \underset{f_t := G_t \circ K}{\overset{\text{[Edwards-Kirby] iso. extension theorem}}{\Longleftrightarrow}} \boxed{\text{Ambient isotopy}} \underset{???}{\overset{f := G_1}{\Longleftrightarrow}} \boxed{\text{Equivalence}}$$

Isotopy: $\exists f_t : S^1 \hookrightarrow Y$ s.t. $f_0 = K$, $f_1 = J$

Ambient isotopy: $\exists G_t : Y \to Y$ s.t. $G_0 = id_Y$, $G_1 \circ K = J$

Equivalence: $\exists f : Y \xrightarrow{\cong} Y$ o.p. s.t. $f \circ K = J$

For knots $K, J : S^1 \hookrightarrow Y$,

$$\boxed{\text{Isotopy}} \underset{f_t := G_t \circ K}{\overset{\text{[Edwards-Kirby] iso. extension theorem}}{\rightleftarrows}} \boxed{\text{Ambient isotopy}} \underset{???}{\overset{f := G_1}{\rightleftarrows}} \boxed{\text{Equivalence}}$$

Isotopy: $\exists f_t : S^1 \hookrightarrow Y$ s.t. $f_0 = K$, $f_1 = J$

Ambient isotopy: $\exists G_t : Y \to Y$ s.t. $G_0 = id_Y$, $G_1 \circ K = J$

Equivalence: $\exists f : Y \xrightarrow{\cong} Y$ o.p. s.t. $f \circ K = J$

??? : goal of this talk

Def$\stackrel{n}{=}$: $\mathrm{Mod}(Y) := \{ f : Y \xrightarrow{\cong} Y \text{ o.p.} \} / \text{isotopy}$

i.e. $f \sim g$ if $\exists\, G_t : Y \to Y$ s.t. $G_0 = f$, $G_1 = g$

mapping class group of Y

(sometimes the homeotopy gp)

Def$\underline{\underline{n}}$: $\mathrm{Mod}(Y) := \{ f : Y \xrightarrow{\cong} Y \text{ o.p.} \} / \text{isotopy}$

i.e. $f \sim g$ if $\exists\, G_t : Y \to Y$ s.t. $G_0 = f$, $G_1 = g$

mapping class group of Y

(sometimes the homeotopy gp)

[Fisher 1960] $\mathrm{Mod}(S^3) = \{ id \}$

$\underline{\text{Def}^{\underline{n}}}$: $\mathrm{Mod}(Y) := \{ f : Y \xrightarrow{\cong} Y \text{ o.p.} \} / \text{isotopy}$

i.e. $f \sim g$ if $\exists\, G_t : Y \to Y$ s.t. $G_0 = f,\ G_1 = g$

mapping class group of Y

(sometimes the homeotopy gp)

[Fisher 1960] $\mathrm{Mod}(S^3) = \{ id \}$

More generally:

$$\begin{tikzcd}
\boxed{\text{Ambient isotopy}} \arrow[r, Rightarrow, shift left, "f := G_1"] & \boxed{\text{Equivalence}} \arrow[l, Rightarrow, shift left, "\mathrm{Mod}(Y) = \{id\}"]
\end{tikzcd}$$

$\exists\, G_t : Y \to Y$ s.t. $G_0 = id_Y$, $G_1 \circ K = J$

$\exists\, f : Y \xrightarrow{\cong} Y$ o.p s.t. $f \circ K = J$

Def$^{\underline{n}}$: $\mathrm{Mod}(Y) := \{ f : Y \xrightarrow{\cong} Y \text{ o.p.} \} / \text{isotopy}$

i.e. $f \sim g$ if $\exists\, G_t : Y \to Y$ s.t. $G_0 = f,\ G_1 = g$

mapping class group of Y

(sometimes the homeotopy gp)

[Fisher 1960] $\mathrm{Mod}(S^3) = \{id\}$

More generally:

$$\begin{tikzcd}[column sep=huge] \boxed{\text{Ambient isotopy}} \arrow[r, Rightarrow, shift left=2, "f := G_1"] & \boxed{\text{Equivalence}} \arrow[l, Rightarrow, shift left=2, "\mathrm{Mod}(Y)=\{id\}"] \end{tikzcd}$$

Ambient isotopy: $\exists\, G_t : Y \to Y$ s.t. $G_0 = id_Y$, $G_1 \circ K = J$

Equivalence: $\exists\, f : Y \xrightarrow{\cong} Y$ o.p. s.t. $f \circ K = J$

e.g. $\mathrm{Mod}(\text{Poincaré homology sphere}) = \{id\}$ [Boileau-Otal 1991]

Q: Does equivalence $\Longrightarrow$ (ambient) isotopy for knots in general Y^3?

Q : Does equivalence $\Rightarrow$ (ambient) isotopy for knots in general Y^3?

Mapping class groups of 3-mflds are not in general trivial

e.g. $\mathrm{Mod}(S^1 \times S^2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ [Gluck 1962]

Q : Does equivalence $\Longrightarrow$ (ambient) isotopy for knots in general Y^3?

Mapping class groups of 3-mflds are not in general trivial

e.g. $\mathrm{Mod}(S^1 \times S^2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ [Gluck 1962]

generated by

(i) $R : S^1 \times S^2 \longrightarrow S^1 \times S^2$ (o.p.)

$(z, s) \longmapsto (\bar{z}, -s)$

Q : Does equivalence $\Longrightarrow$ (ambient) isotopy for knots in general Y^3?

Mapping class groups of 3-mflds are not in general trivial

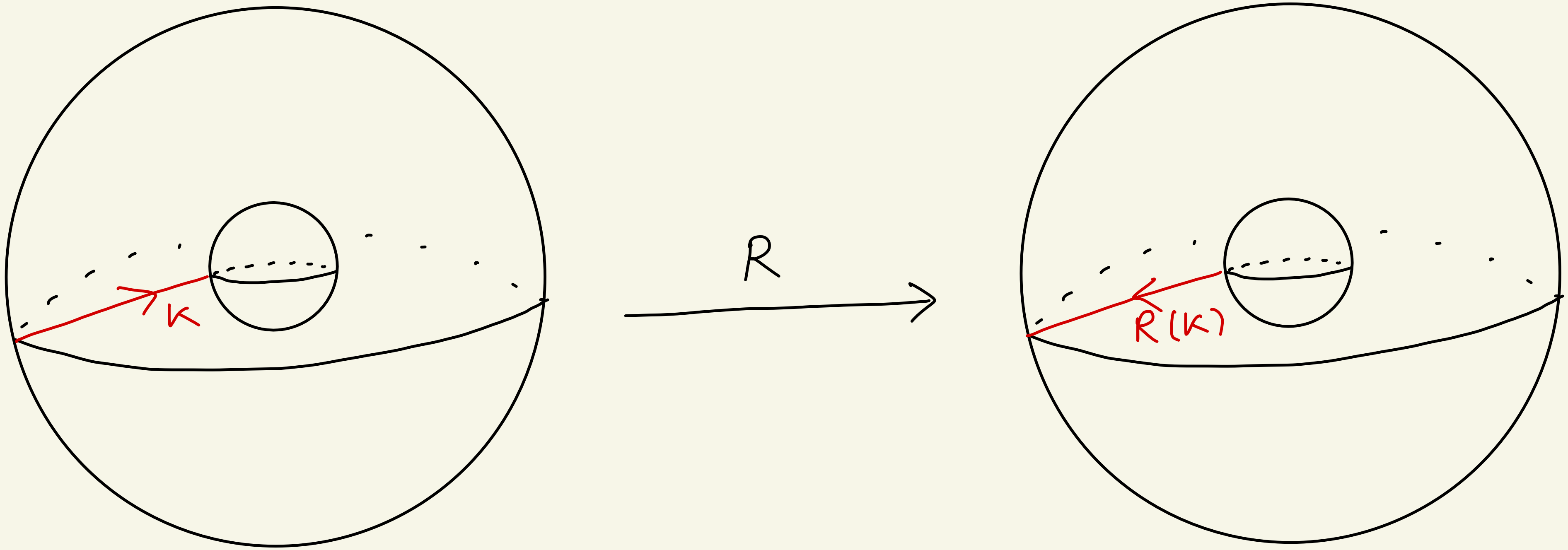
e.g. $\mathrm{Mod}(S^1 \times S^2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ [Gluck 1962]

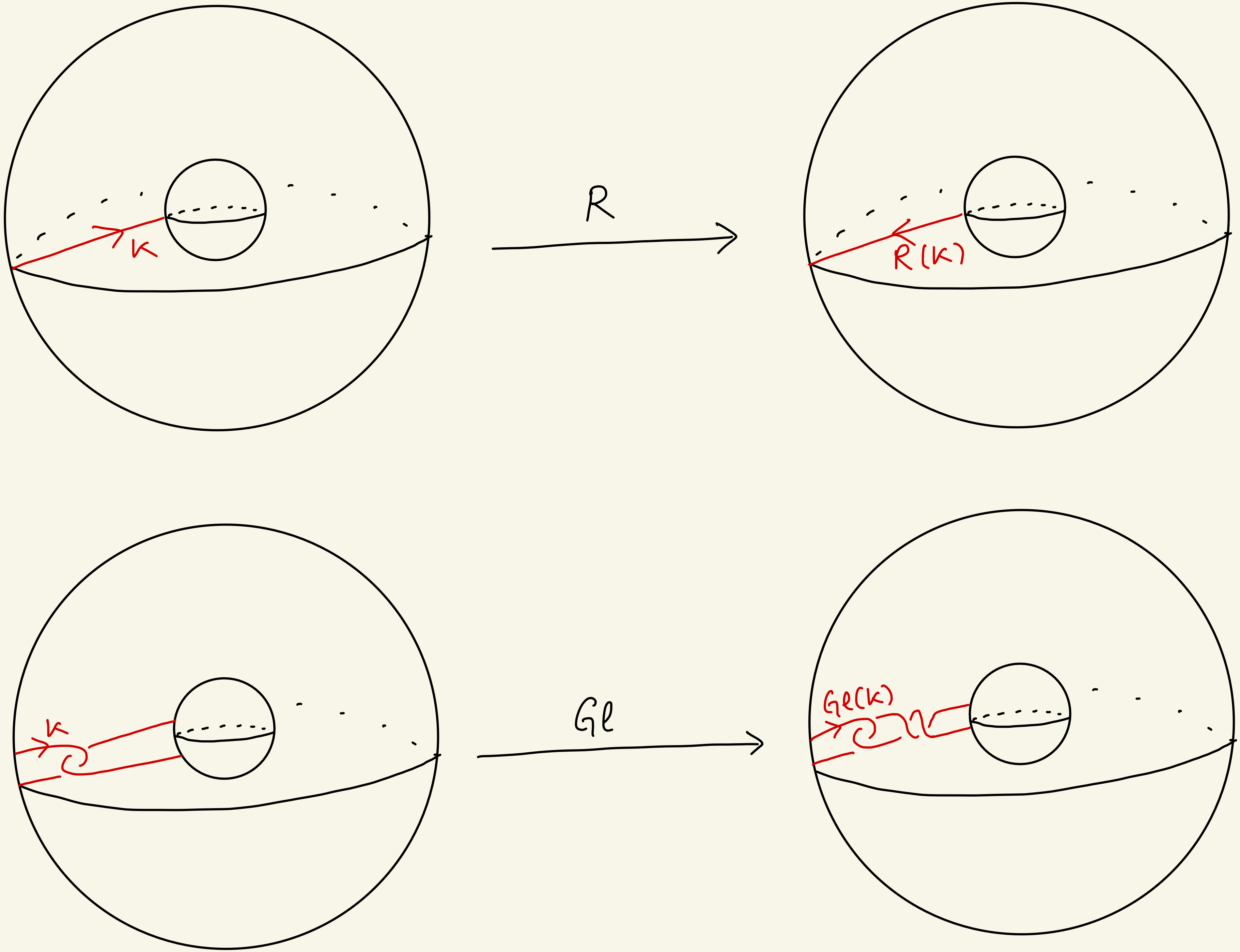
generated by

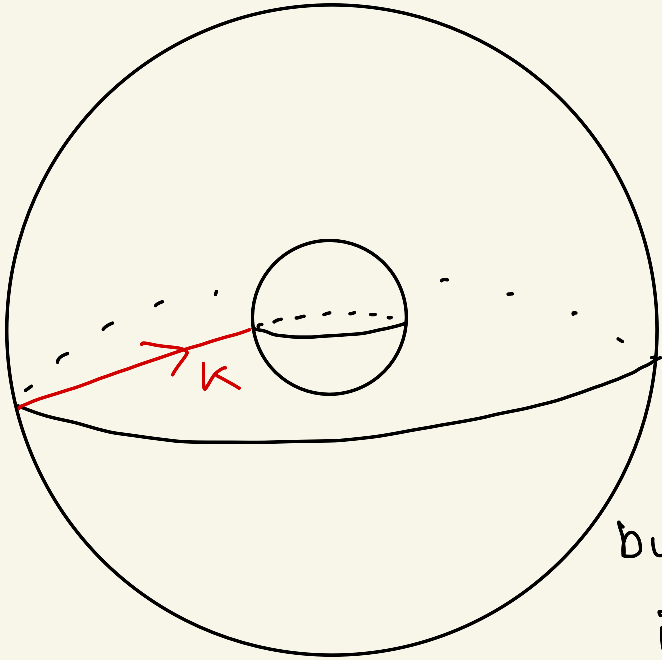
(i) $R : S^1 \times S^2 \longrightarrow S^1 \times S^2$ (o.p.)
$\quad (z, s) \longmapsto (\bar{z}, -s)$

(ii) $G : S^1 \times S^2 \longrightarrow S^1 \times S^2$ Gluck twist
$\quad (\theta, s) \longmapsto (\theta, \rho_\theta(s))$

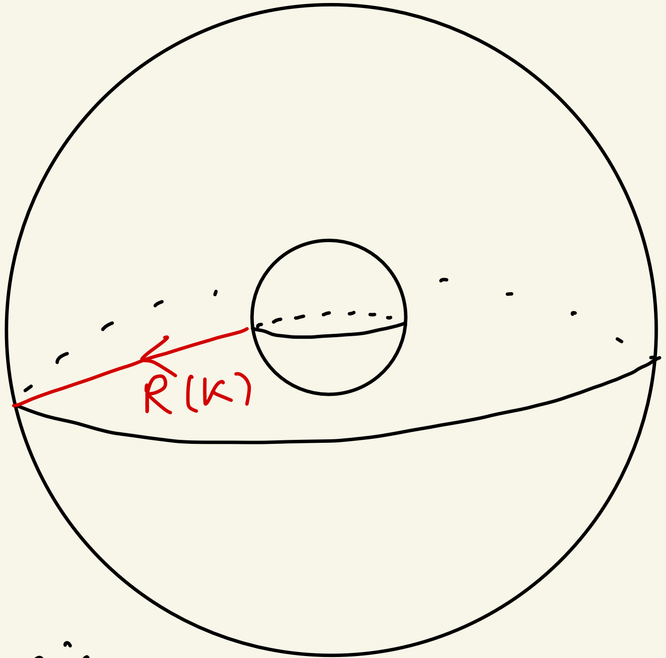
rotate by θ about fixed axis.


K
R
R(K)


K
R
R(K)
K
Gl
Gl(K)



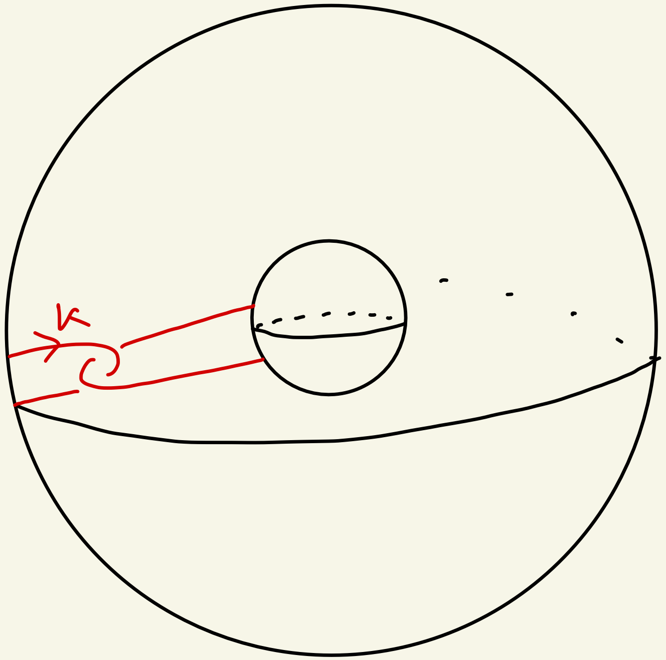
$\xrightarrow{R}$



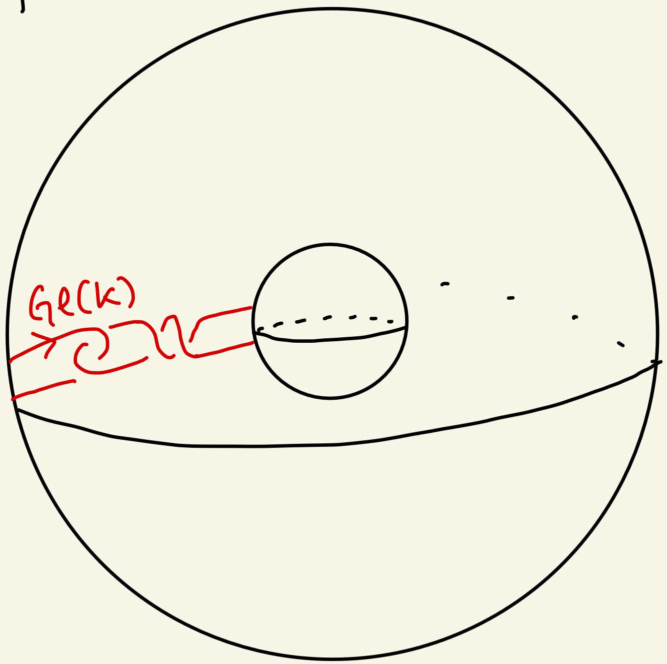
K, R(K) equiv.

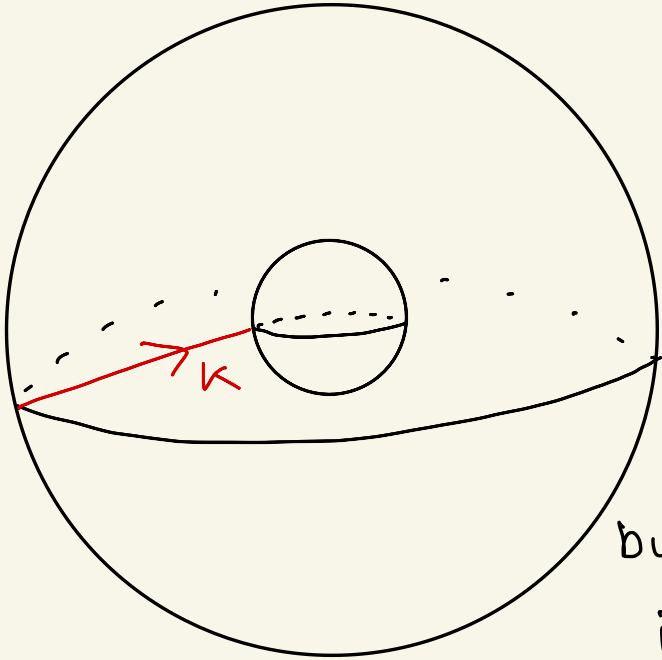
but $[R(K)] = -[K]$ in $\pi_1(S^1 \times S^2)$

$\Rightarrow$ K, R(K) not isotopic or even homotopic

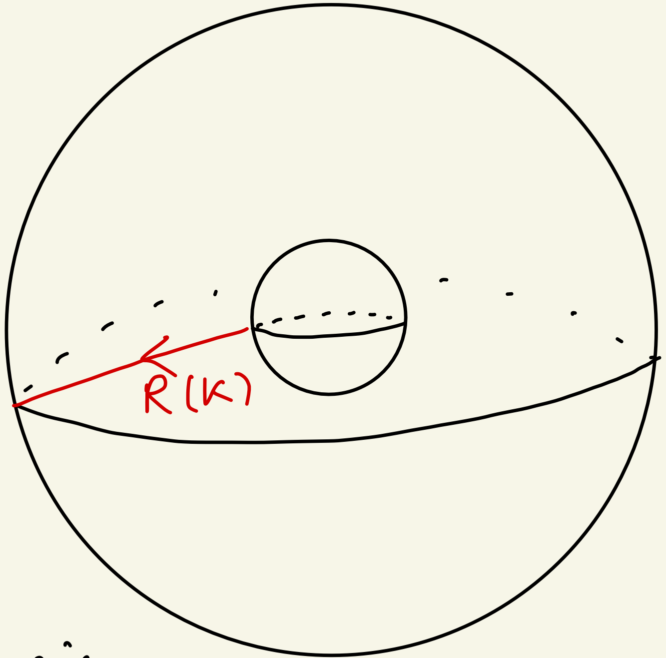


$\xrightarrow{Gl}$





$\xrightarrow{R}$



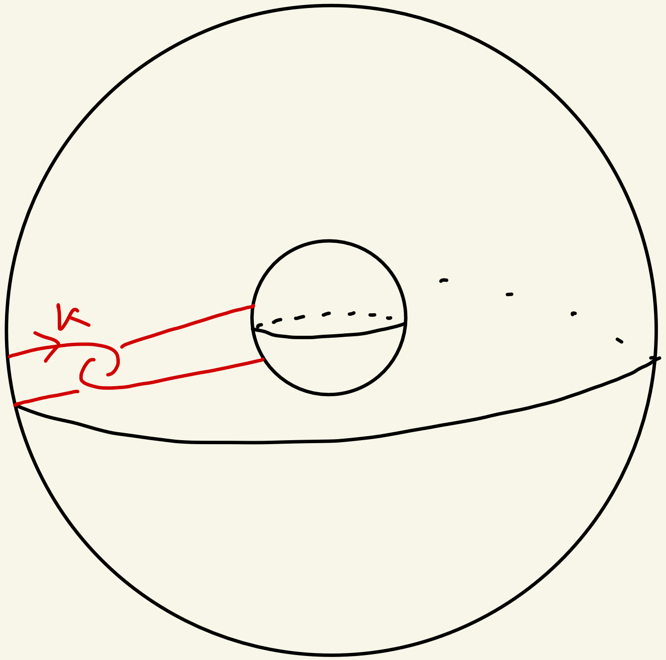
K, R(K) equiv.

but $[R(K)] = -[K]$

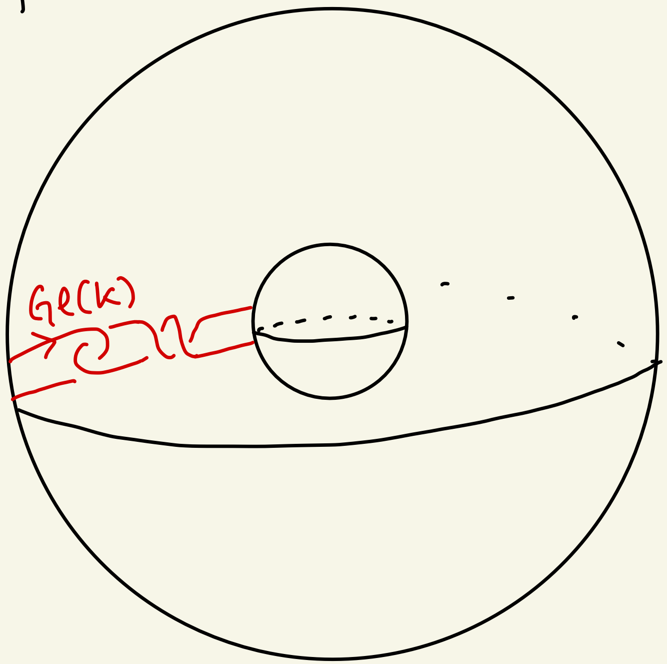
in $\pi_1(S^1 \times S^2)$

$\Rightarrow$ K, R(K) not isotopic

or even homotopic



$\xrightarrow{Gl}$



K, Gl(K) equiv.

Gl preserves free

homotopy classes

of loops

Theorem 1 [Aceto - Bregman - Davis - Park - R.]

For every winding number $w \in \mathbb{Z}$, $\exists\ K_w : S^1 \hookrightarrow S^1 \times S^2$

s.t. K_w and $Gl(K_w)$ are not isotopic.

Indeed, K_w is not isotopic to $Gl(K_w)$ for any K_w

with w odd, $\neq \pm 1$.

More generally:

<u>Def$^{\underline{n}}$</u>: G a group. An automorphism $\varphi : G \xrightarrow{\cong} G$ is **class preserving** if it fixes every conjugacy class

More generally:

<u>Defⁿ</u>: G a group. An automorphism $\varphi : G \xrightarrow{\cong} G$ is **class preserving** if it fixes every conjugacy class

G has **Grossman's Property A** if every class preserving automorphism is inner

More generally:

<u>Def$^{\underline{n}}$</u>: G a group. An automorphism $\varphi: G \xrightarrow{\cong} G$ is **class preserving** if it fixes every conjugacy class

G has **Grossman's Property A** if every class preserving automorphism is inner

e.g. (abelian gps), free gps, surface gps [Grossman 1974]

torsion free hyperbolic gps, [Minasyan-Osin 2010]

rel. hyperbolic gps

nontrivial free products [Neschadim 1996]

More generally:

Def$^{\underline{n}}$: G a group. An automorphism $\varphi : G \xrightarrow{\cong} G$ is **class preserving** if it fixes every conjugacy class

G has **Grossman's Property A** if every class preserving automorphism is inner

e.g. (abelian gps), free gps, surface gps [Grossman 1974]

torsion free hyperbolic gps, [Minasyan-Osin 2010]

rel. hyperbolic gps

nontrivial free products [Neschadim 1996]

$\exists$ nilpotent gps which do not enjoy Property A

Theorem 2 [ABDPR]: Y^3 closed, orientable.

Then $\pi_1(Y)$ enjoys Property A

Theorem 2 [ABDPR]: Y^3 closed, orientable.
Then $\pi_1(Y)$ enjoys Property A

Remark: known in many cases by work of
[Antolin-Minasyan-Sisto 2016]
[Allenby-Kim-Tang 2003, 2009]

We give an alternative proof via a topological approach and treat the remaining cases.

Corollary 3 [ABDPR]: Y^3 closed, oriented, prime

$f: Y \to Y$ o.p. homeo with $f(K)$ and K freely homotopic $\forall K$

Then either (i) f is isotopic to id_Y

(ii) $Y = S^1 \times S^2$ and $f = Gl$.

Corollary 3 [ABDPR]: Y^3 closed, oriented, prime

$f: Y \to Y$ o.p. homeo with $f(K)$ and K freely homotopic $\forall K$

Then either (i) f is isotopic to id_Y

(ii) $Y = S^1 \times S^2$ and $f = Gl$.

Proof: Y irred $\Longrightarrow$ $Mod(Y) \hookrightarrow Out(\pi_1 Y)$

by Hatcher, Waldhausen, Gabai-Meyerhoff-Thurston, Scott, Boileau-Otal, ...

Corollary 3 [ABDPR]: Y^3 closed, oriented, prime

$f: Y \to Y$ o.p. homeo with $f(K)$ and K freely homotopic $\forall K$

Then either (i) f is isotopic to id_Y

(ii) $Y = S^1 \times S^2$ and $f = Gl$.

Proof: Y irred $\Longrightarrow$ $Mod(Y) \hookrightarrow Out(\pi_1 Y)$

by Hatcher, Waldhausen, Gabai-Meyerhoff-Thurston, Scott, Boileau-Otal, ...

$[K] = [f(K)]$ $\forall K$ $\Rightarrow$ f is class preserving

$\xRightarrow{\text{Thm 2}}$ f is inner $\Rightarrow$ trivial in $Out(\pi_1 Y)$

$\Rightarrow$ trivial in $Mod(Y)$.

$\square$

Corollary 4 [ABDPR]: Y closed, oriented, prime

$f: Y \to Y$ o.p. homeo. If $f(K)$ is isotopic to K $\forall K$

then f is isotopic to id_Y.

i.e. equivalent knots in Y are isotopic

$$\Updownarrow$$

$$Mod(Y) = \{id_Y\}$$

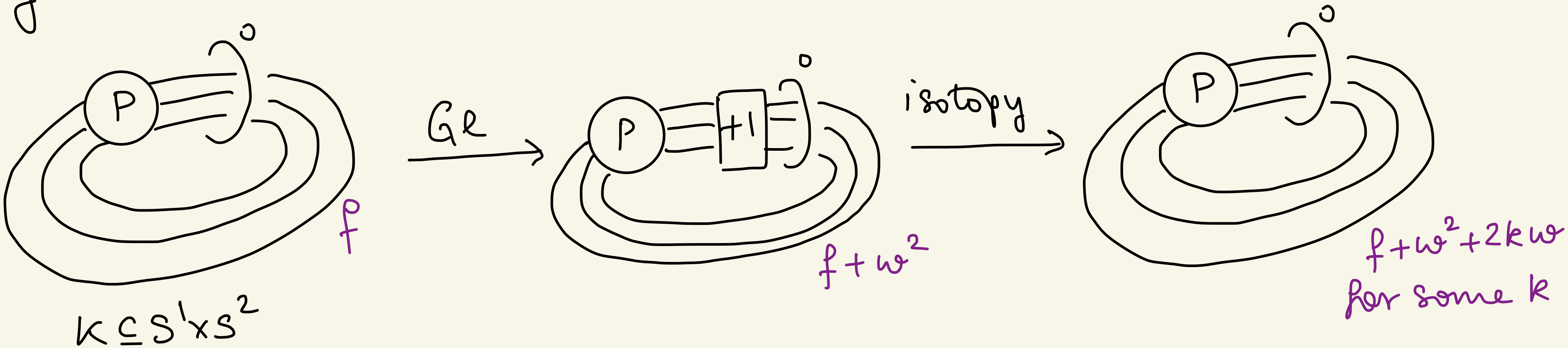
Recall:

Theorem 1 [ABDPR]:

For every winding number $w \in \mathbb{Z}$, $\exists\, K_w : S^1 \hookrightarrow S^1 \times S^2$ s.t. K_w and $Gl(K_w)$ are not isotopic

Indeed, K_w is not isotopic to $Gl(K_w)$ for any K_w with w odd, $\neq \pm 1$

Key lemma:



[McCullough 2006] if $M(D,f) \cong M(D, f+w^2+2kw)$

then either • K has geometric winding number ± 1

• $w^2+2kw=0$

w odd, $\neq \pm 1$: done.

w even, $\neq 0$: $\exists$ nontrivial homeo $M(D,f) \xrightarrow{\cong} M(D,f)$

- obstruct using d-invariants $w \neq \pm 2$

SnapPy, Sage $w = \pm 2$

$w=0$: use satellite construction

+ uniqueness of JSJ decomposition

Thanks for you attention!