A surface embedding theorem

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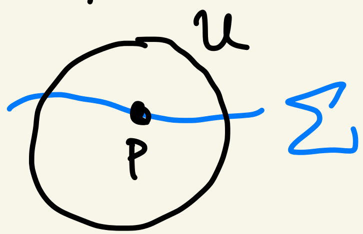
Q: Given a map of a surface in a 4-mfld, when is it homotopic to a (loc. flat) embedding?



$$(U, U \cap \Sigma) \overset{\text{homeo}}{\approx} (\mathbb{R}^4, \mathbb{R}^2)$$

e.g.
- which elements of $\pi_2(M^4)$ are rep. by embedded spheres?
- when does a knot in S^3 bound an embedded disc in B^4?

Prototypical result: Disc embedding theorem

Casson, Freedman'82,
Freedman-Quinn'90

M^4 connected topological mfld, $\pi_1 M$ good.

$\Sigma = \sqcup \Sigma_i$ compact surface, each Σ_i simply connected

$$\begin{tikzcd} F:\Sigma \arrow[r, looparrowright] & M \\ \partial\Sigma \arrow[u, hook] \arrow[r, hook] & \partial M \arrow[u, hook] \end{tikzcd}$$

a generic immersion

such that • the algebraic intersection numbers of F vanish

• $\exists G: \sqcup S^2 \looparrowright M$ framed, alg. dual spheres for F

Then F is (reg.) htpic rel ∂ to a loc. flat embedding $\bar{F}$

[with geom dual spheres $\bar{G}$ s.t. $\bar{G} \simeq G$.] $\pi_1 \neq 1$ Powell-R-Teichner'20.

Generic immersions

- locally an (loc. flat) embedding, intersections isolated double points (2+2=4)

- any continuous map $\Sigma^2 \longrightarrow M^4$ is htpic to a gen. imm.

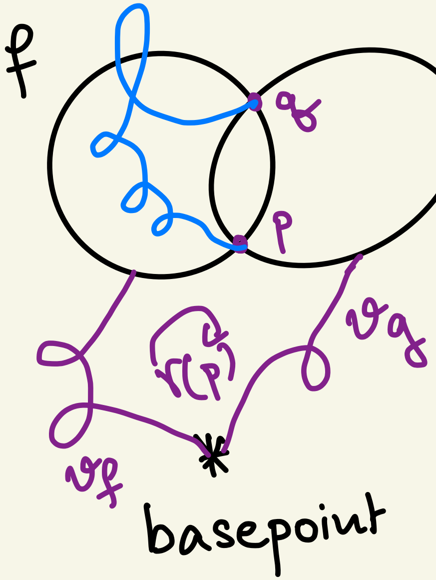
  [FQ ; see PRT '20]

  uses that any noncompact, connected 4-mfld is smoothable. [Quinn]

Good groups

- abelian gps, finite gps, solvable gps, ...
- gps of subexp growth
- closed under subgps, quotients, extensions, direct limits
- open whether all gps are good e.g. $\mathbb{Z} * \mathbb{Z}$?

Intersection numbers



$$\lambda(f,g) := \sum_{p \in f \pitchfork g} \varepsilon(p)\,\gamma(p) \quad \in \mathbb{Z}[\pi_1 M]$$

$\varepsilon(p)$: sign of int p.

well defined if f, g are simp. connected (modulo whiskers)

$\lambda(f,g)=0 \iff$ all points in $f \pitchfork g$ are paired by gen. immersed discs, with (framed) disjoint, embedded boundaries

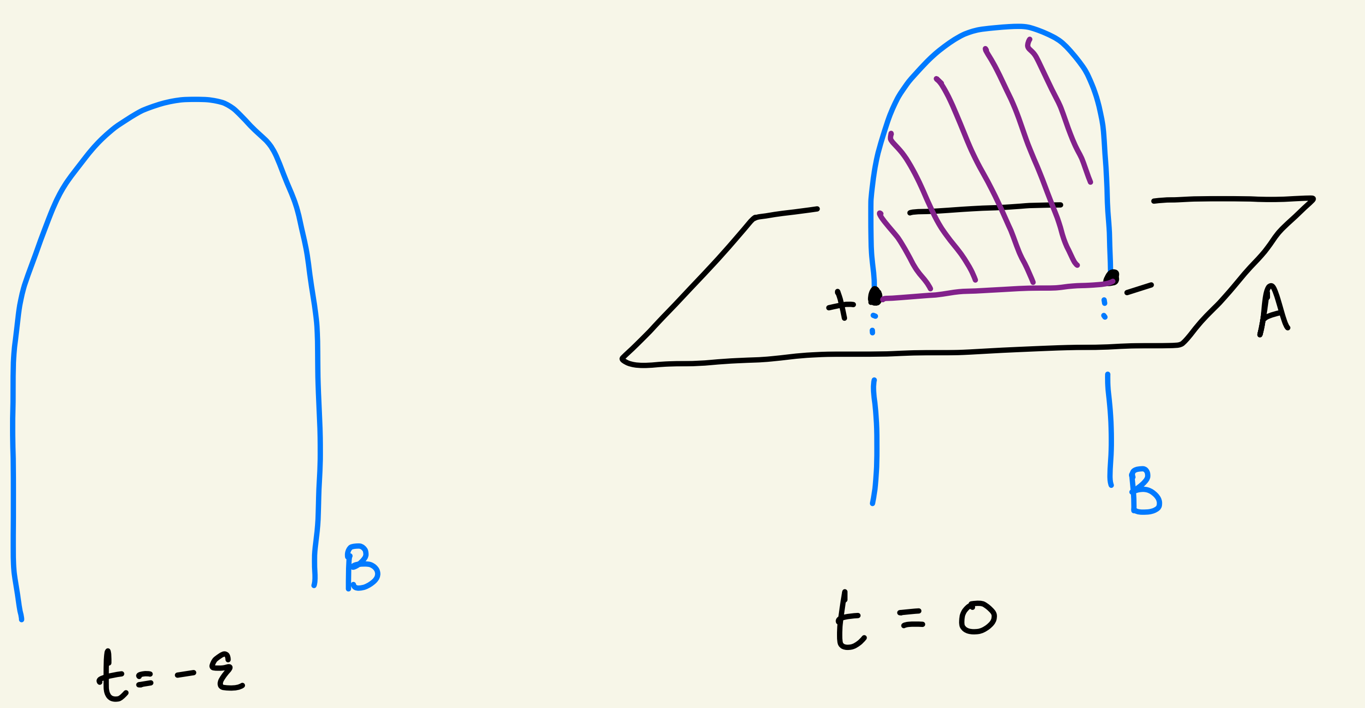
gen coll. of Wh discs

Self-intersection number $\mu(f)=0 \iff$ all pts in $f \pitchfork f$ are paired by gen coll. of Wh discs

f, g are alg dual if $\lambda(f,g)=1 \iff$ all but one pt in $f \pitchfork g$ are so paired

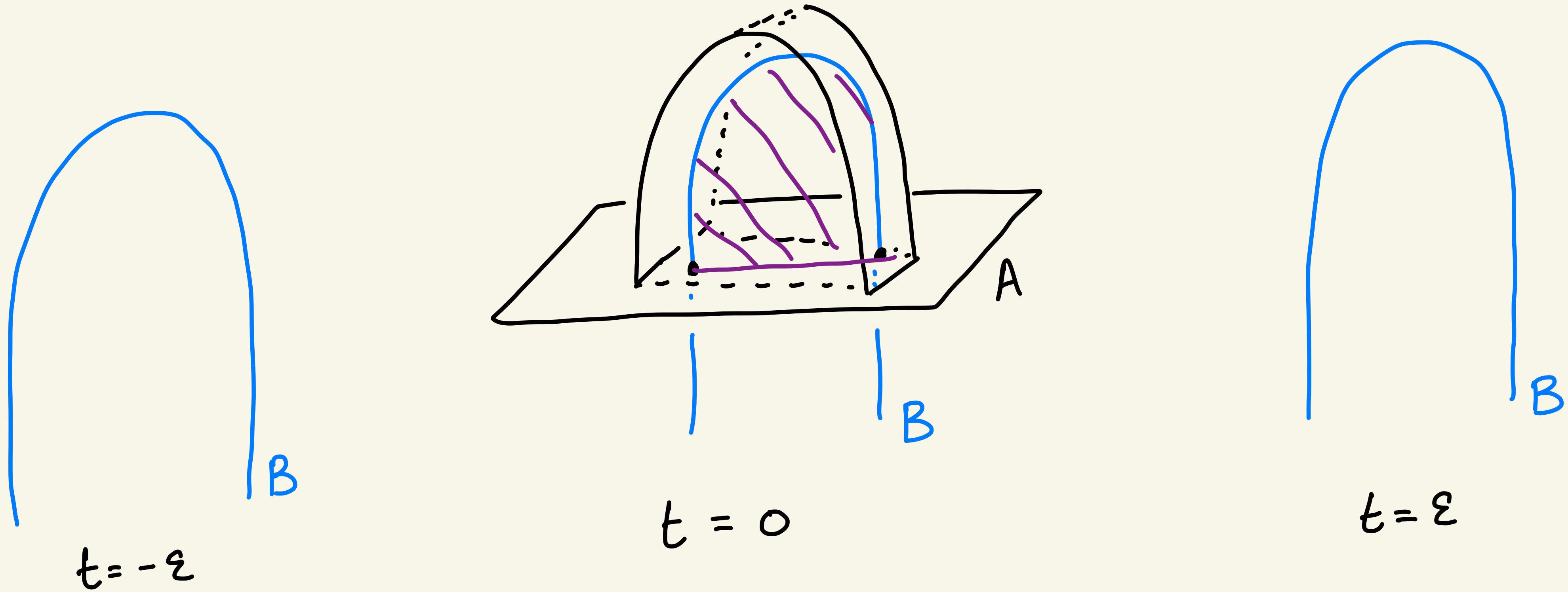
f, g are geom dual if $f \pitchfork g = \{pt\}$

The Whitney trick



$\mathbb{R}^4 \cong \mathbb{R}^3 \times \{\text{time}\}$

The Whitney trick



~~Disc~~ Surface embedding theorem — Casson, Freedman '82, Freedman-Quinn '90, Stong '94, Kasprowski-Powell-R-Teichner 21+

M^4 connected topological mfld, $\pi_1 M$ good.

$\Sigma = \sqcup \Sigma_i$ compact surface, each Σ_i ~~simply~~ connected

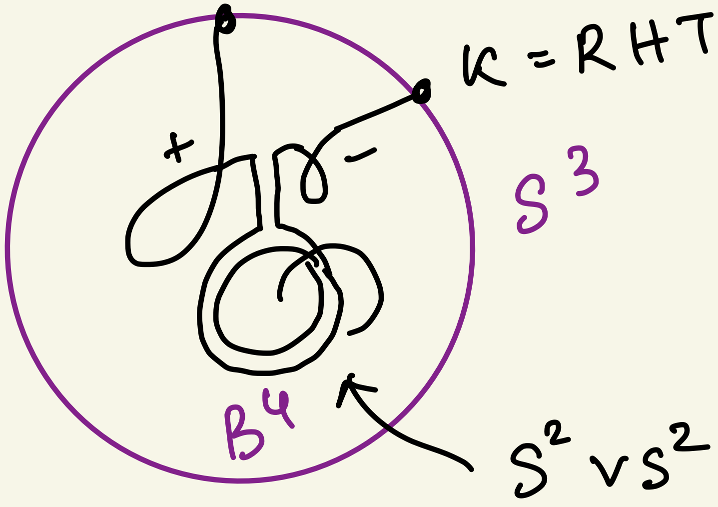
$$\begin{tikzcd} \Sigma \arrow[r, "F", looparrowright] & M \\ \partial\Sigma \arrow[u, hook] \arrow[r, hook] & \partial M \arrow[u, hook] \end{tikzcd}$$

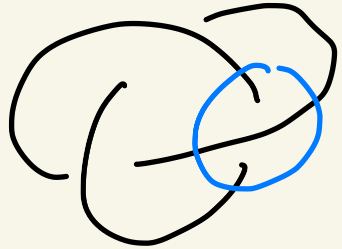
a generic immersion

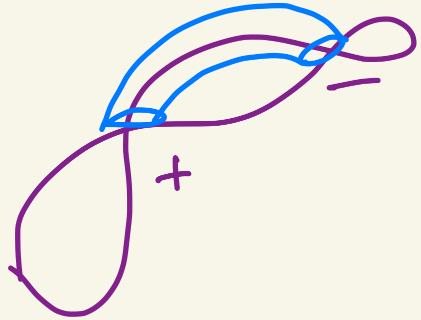
such that
- algebraic intersection numbers of F vanish
- $\exists\, G : \sqcup S^2 \looparrowright M$ ~~framed~~, alg. dual spheres for F

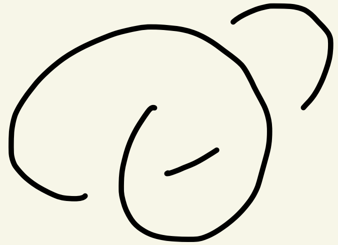
Then F is reg. htpic rel ∂ to a loc. flat embedding $\bar{F}$ with geom dual spheres $\bar{G}$ s.t. $\bar{G} \simeq G$.

if and only if the Kervaire-Milnor invariant $km(F) \in \mathbb{Z}/2$ vanishes









RHT is not slice   i.e. $\nexists$ emb disc bounded by K.

Every $K \subseteq S^3$ bounds an emb. disc in $\#_n \mathbb{CP}^2 \#_m \overline{\mathbb{CP}^2}$

given K, min m s.t. K null-hom slice in $\#_m \mathbb{CP}^2$

$- \# S^2 \tilde{\times} S^2$

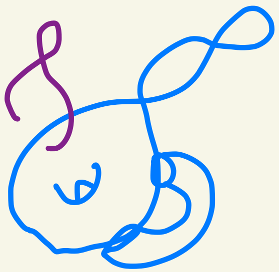
$S^2 \times S^2$

[nullhom. disc in $\# S^2 \times S^2$ iff $\mathrm{Arf}(K) = 0$]

Corollary 1 : $F : \Sigma^2 \looparrowright M^4$ with

$\pi_1 M$ good.

- Σ connected
- alg int numbers vanish
- $\exists$ G alg. dual sphere

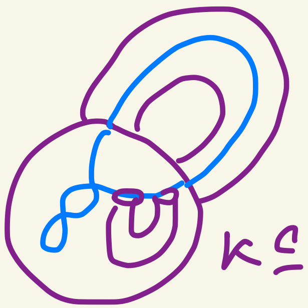


$F' :=$ result of adding a trivial tube to F

Then F' is (reg) htpic to an embedding

Corollary 2: $F : \Sigma^2 \looparrowright M^4$ with

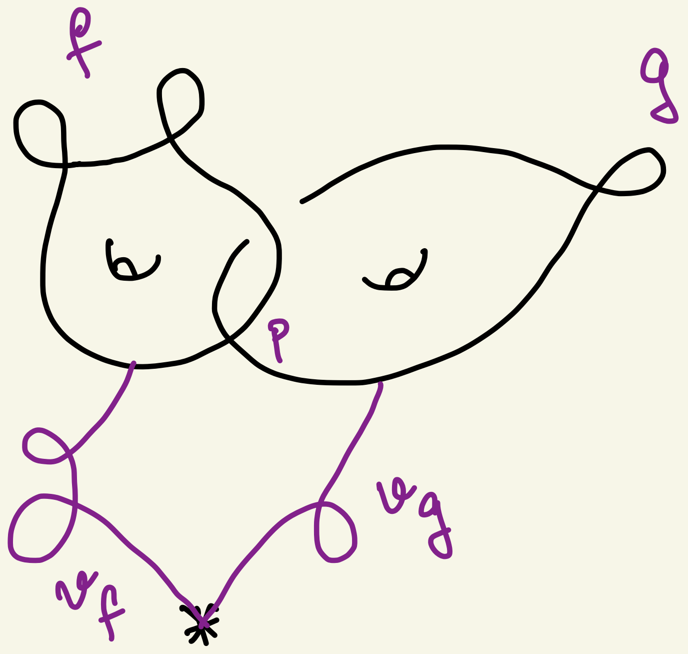
- Σ connected, $g(\Sigma) > 0$
- alg int numbers vanish
- $\exists$ G alg dual sphere
- $\pi_1 M = 1$ {trivial gp is good}



Then F is (reg) htpic to an embedding

Corollary [FMNOPR] $g_{sh}^{top, \pm 1}(K) \leq 1$.

Intersection numbers



$\lambda(f,g)$ not well defined in $\mathbb{Z}[\pi_1 M]$!

but count in a double coset space

$\lambda(f,g)=0 \iff$ all pts in $f \pitchfork g$ paired by gen imm coll of Wh discs

$\mu(f)=0 \iff$ all pts in $f \pitchfork f$ paired by gen imm coll of Wh discs

The Kervaire-Milnor invariant

[for discs/spheres, due to FQ90 §10 + Stong]

$\Sigma = \sqcup \Sigma_i$

$F:\Sigma \looparrowright M$ trivial alg int numbers, $\exists\, G: \sqcup S^2 \looparrowright M$ alg dual

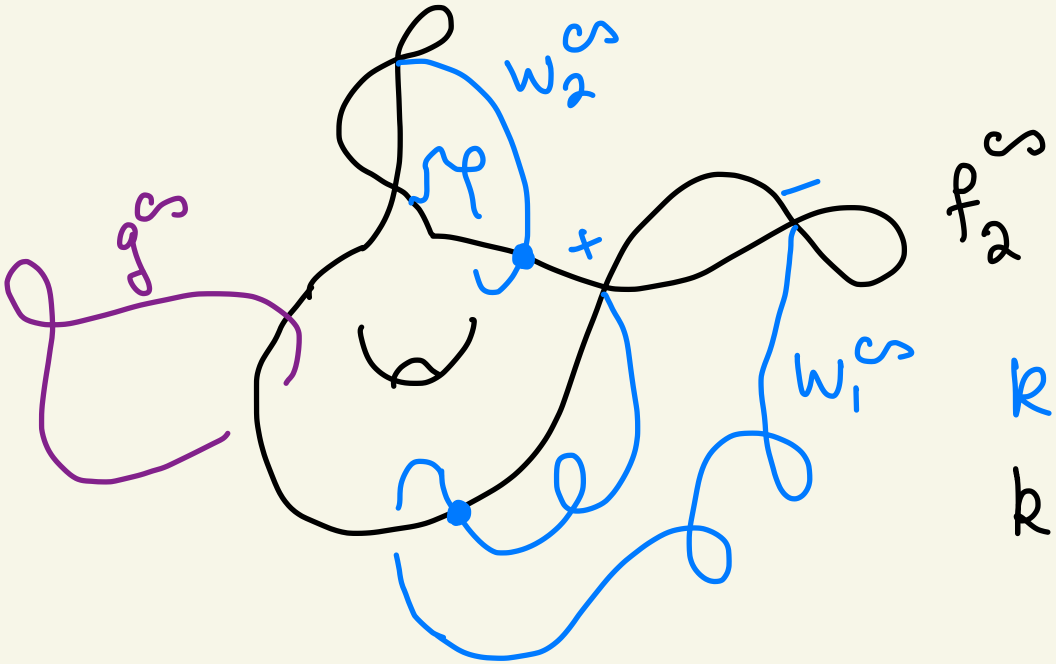
$\Rightarrow f \pitchfork f$ are paired by gen. coll. of Wh discs $\mathcal{W}$

Let $\Sigma^{cs} \subseteq \Sigma$ subsurface, $F^{cs} := F|_{\Sigma^{cs}}$ admits only twisted duals

i.e. euler number of the norm. bundles are odd.

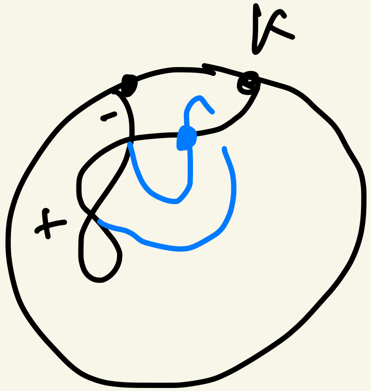
Let $\mathcal{W}^{cs} = \{W_\ell^{cs}\} \subseteq \mathcal{W}$ subset pairing ints of F^{cs}.

Then $km(F, \mathcal{W}) := \sum_\ell |\operatorname{Int} W_\ell^{cs} \pitchfork F^{cs}| \bmod 2.$



$km(f_1^{cs}, w) = 1$

$km(f_2^{cs}, w) = 0 \in \mathbb{Z}/2$



← Arf(K)

Question: When is $km(F, \mathcal{W})$ independent of $\mathcal{W}$?

(spoiler : when F is b-characteristic)

Proof outline: Suppose $\exists \mathcal{W}$ s.t. $\mathrm{km}(F, \mathcal{W}) = 0 \in \mathbb{Z}/2$

Step 1: By reg. htpy, make F and G geom dual (still immersed)
(standard trick)

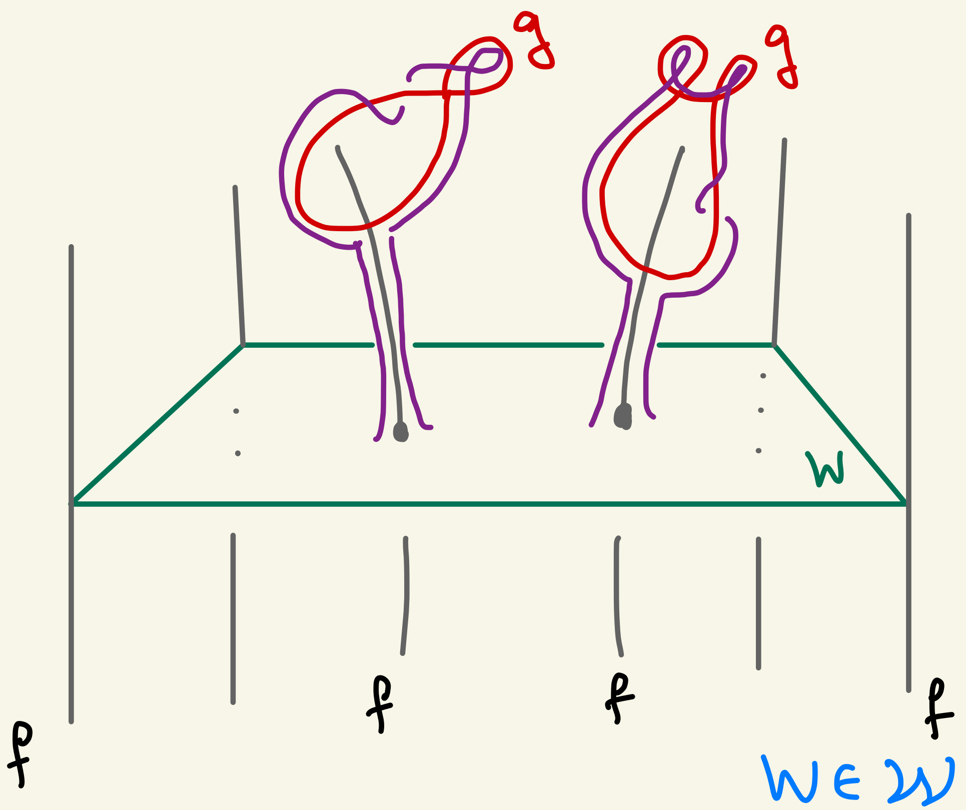
Step 2: Upgrade $\mathcal{W}$ and F by reg htpy s.t. $\{\mathrm{Int}\, W_e\} \cap F = \emptyset$

Step 3: Use (Whitney) disc embedding theorem
to replace $\mathcal{W}$ by $\{V_e\}$ s.t.
- $\{\mathrm{Int}\, V_e\} \cap F = \emptyset$
- $\{V_e\}$ flat, embedded, disjoint
- $\exists$ geom dual spheres $\{V_e^T\}$ in $M \setminus F$

Step 4: Tube G into $\{V_e^T\}$ to get $\bar{G}$, geom dual to F, disjoint from $\{V_e\}$

Step 5: Whitney move F over $\{V_e\}$ to get desired F.

Step 2: Upgrade $\mathcal{W}$ and F by neg htpy s.t. $\{\mathrm{Int}\, W_e\} \cap F = \emptyset$



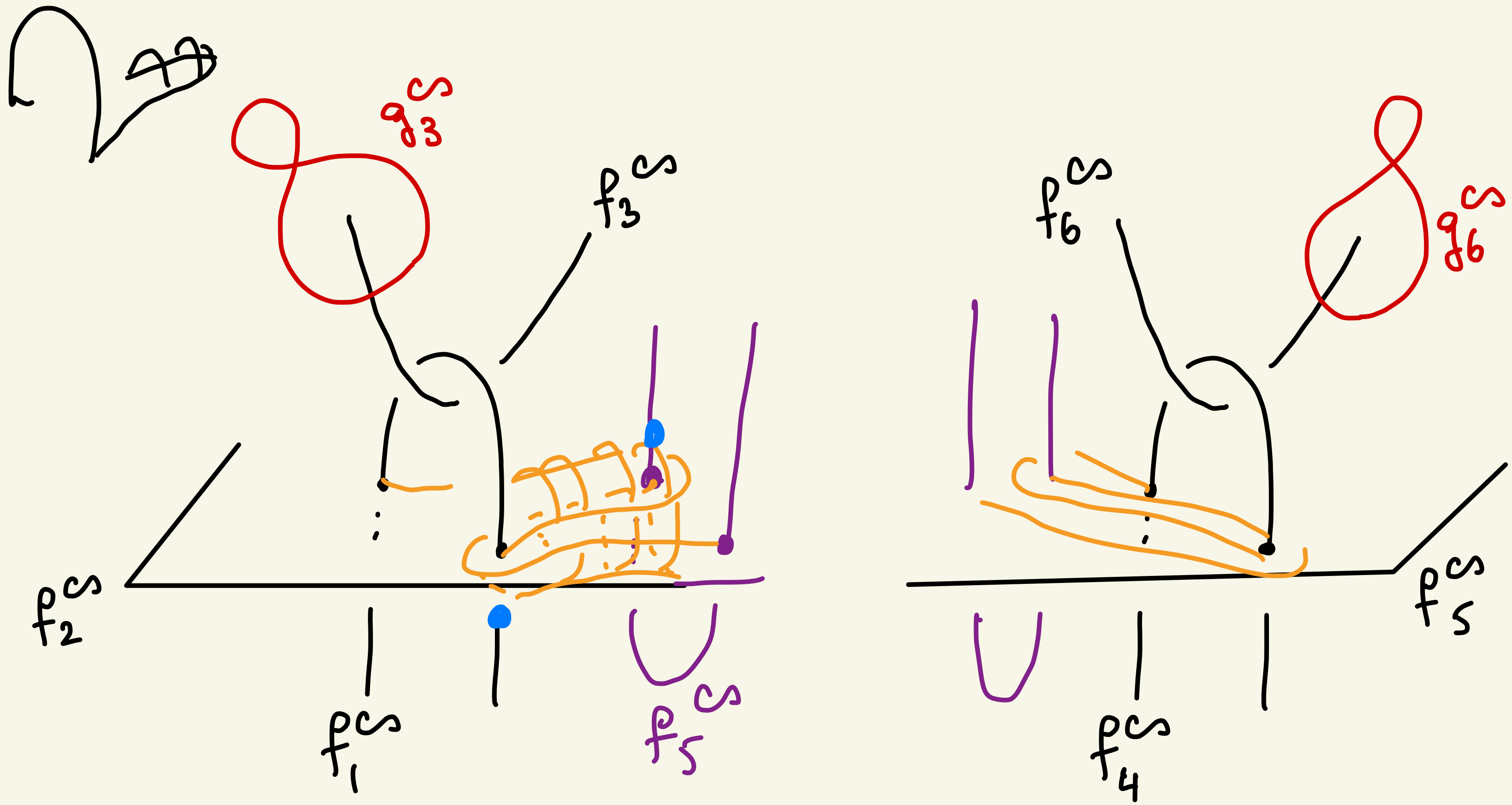
g	framing of W
not twisted	change by even number
twisted	change by odd number

- local cusp moves in Int W changes framing by ± 2.

Step 2: Upgrade W and F by reg htpy s.t. $\{\mathrm{Int}\, W_e\} \cap F = \emptyset$

Remaining problem: Wh discs for F with a single "problem" each

$km = 0$ $\rightarrow$ there are even such problem discs.



"transfer" move

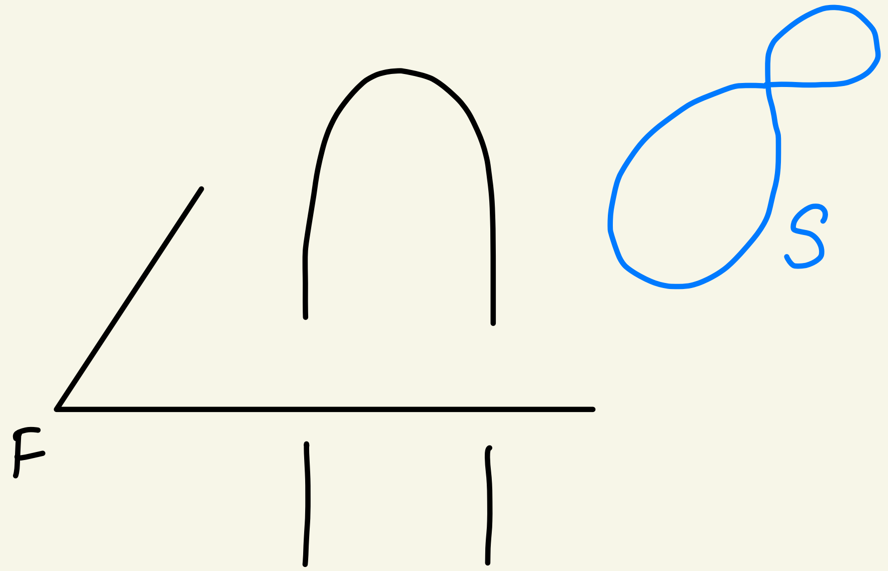
• do a finger move between f_2^{cs} and f_5^{cs}

Thanks!

When is $km(F, \mathcal{W})$ independent of $\mathcal{W}$?

- For convenience, assume Σ connected; M, Σ oriented; $\Sigma = \Sigma^{cs}$

Suppose $\exists$ immersed sphere $S: S^2 \looparrowright M$
s.t. $F \cdot S \not\equiv S \cdot S \mod 2$



(i) $F \cdot S$ odd
$S \cdot S$ even

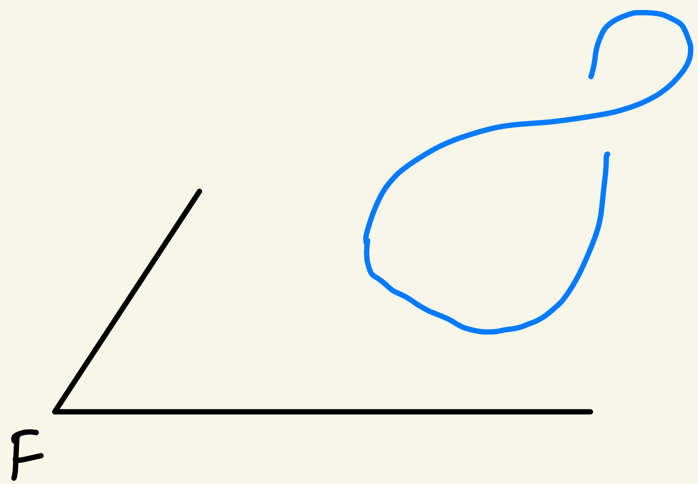
(ii) $F \cdot S$ even
$S \cdot S$ odd

Otherwise, F is called s-characteristic.

When is $km(F, W)$ independent of W?

- For convenience, assume Σ connected; M, Σ oriented; $\Sigma = \Sigma^{cs}$

Suppose $\exists$ immersed $\mathbb{R}P^2$ $R: \mathbb{R}P^2 \looparrowright M$
s.t. $F \cdot R \not\equiv R \cdot R \mod 2$



Otherwise, F is called r-characteristic.

When is $\mathrm{km}(F, \mathcal{W})$ independent of $\mathcal{W}$?

- For convenience, assume Σ connected; M, Σ oriented.

If $\lambda_\Sigma|_{\partial B(F)}$ trivial, for a band B and A a collection of wharcs for F,

$$\Theta_A(B) := \mu_\Sigma(\partial B) + \partial B \pitchfork A + B \pitchfork F + e(B) \bmod 2$$

Suppose $\exists A, B$ s.t. $\Theta_A(B) \neq 0$

When is $\mathrm{km}(F, \mathcal{W})$ independent of $\mathcal{W}$?

- For convenience, assume Σ connected; M, Σ oriented; $\Sigma = \Sigma^{cs}$

Let $\mathcal{B}(F) \subseteq H_2(M, \Sigma; \mathbb{Z}/2)$ the subset rep by maps of annuli or Möbius bands

Suppose the $\mathbb{Z}/2$ int form λ_Σ on $H_1(\Sigma; \mathbb{Z}/2)$ is nontrivial on $\partial \mathcal{B}(F)$.

When is $km(F, \mathcal{W})$ independent of $\mathcal{W}$?

- For convenience, assume Σ connected; M, Σ oriented; $\Sigma = \Sigma^{cs}$

Lemma: $\Theta_A(B)$ depends only on the homology class of B.

If $\lambda_\Sigma|_{\partial B(F)} = 0$, then Θ_A does not depend on A.

$\rightsquigarrow$ $\exists$ well defined $\quad \Theta : B(F) \longrightarrow \mathbb{Z}/2$.

Definition: F is b-characteristic if $\lambda_\Sigma|_{\partial B(F)} = 0$ & $\Theta = 0$

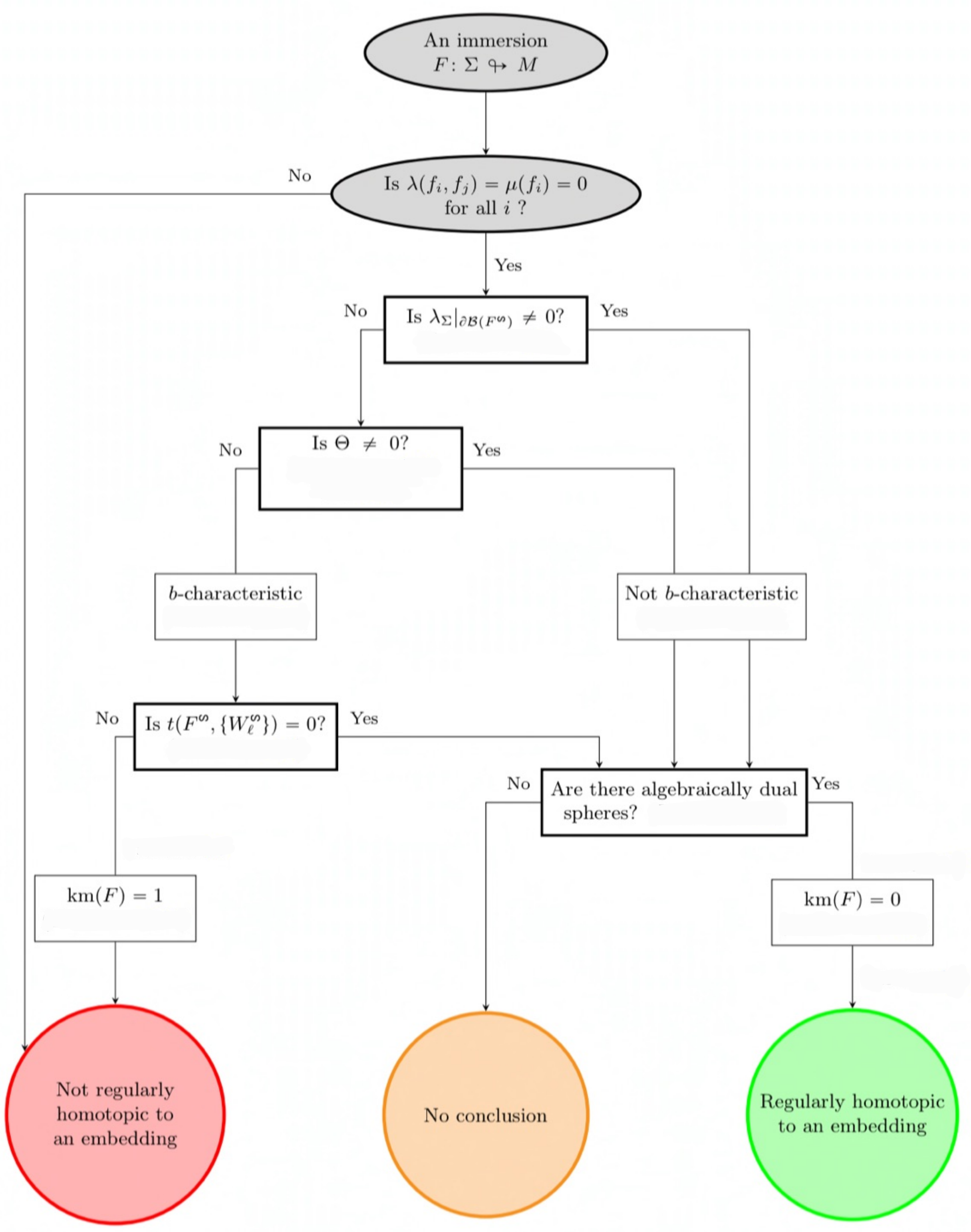
When is $km(F, \mathfrak{w})$ independent of $\mathfrak{w}$?

- For convenience, assume Σ connected; M, Σ oriented; $\Sigma = \Sigma^{cs}$

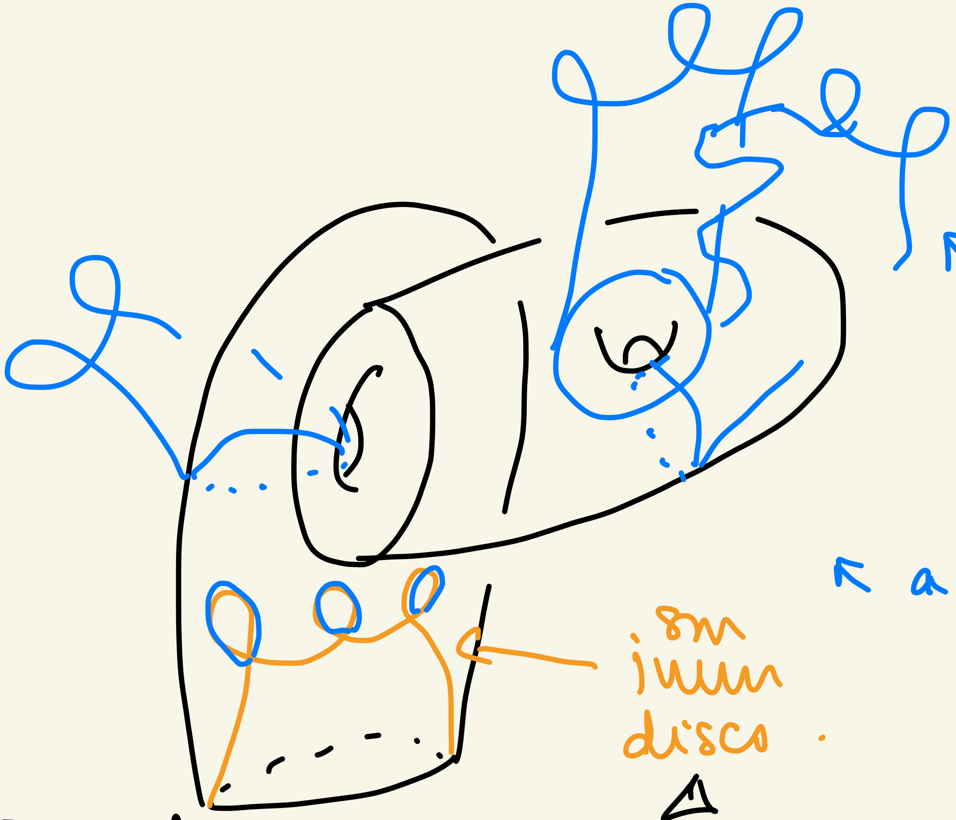
Theorem (KPRT): $km(F, \mathfrak{w})$ is independent of $\mathfrak{w}$ iff F is b-characteristic

Definition: $km(F) = 0$ if F not b-char

$km(F) = km(F, \mathfrak{w})$ if F b-char.



An immersion $F\colon \Sigma \looparrowright M$
No
Is $\lambda(f_i, f_j) = \mu(f_i) = 0$ for all i ?
Yes
No
Is $\lambda_\Sigma|_{\partial\mathcal{B}(F^\omega)} \neq 0$?
Yes
No
Is $\Theta \neq 0$?
Yes
b-characteristic
Not b-characteristic
No
Is $t(F^\omega, \{W_\ell^\omega\}) = 0$?
Yes
No
Are there algebraically dual spheres?
Yes
$\mathrm{km}(F) = 1$
$\mathrm{km}(F) = 0$
Not regularly homotopic to an embedding
No conclusion
Regularly homotopic to an embedding



↖ plumbed $D^2 \times D^2$

↖ a smooth 4-mfld.

sm imm discs.

Δ

Γ is good if

$$\forall \varphi : \pi_1(G(1.5)^c) \longrightarrow \Gamma \quad \exists \text{ disc } \Delta \quad \text{s.t.} \quad \varphi(\partial \Delta) = e$$

π_1-null disc property