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Techtopology 10

Embedding surfaces in 4-manifolds

joint with
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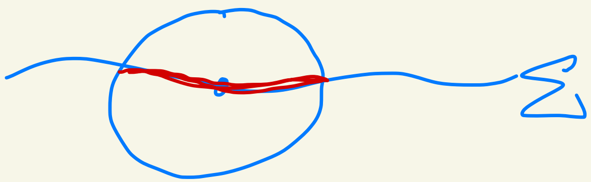
Embedding surfaces in 4-manifolds

(joint w. Kasprowski, Powell, Teichner)

Q: Given a map of a surface in a 4-mfld, when is it homotopic to a (loc. flat or smooth) embedding?

- an embedding $\Sigma \subset M$ is loc. flat if each pt in Σ has a nbd $\mathcal{U}$ s.t. $(\mathcal{U}, \mathcal{U} \cap \Sigma) \underset{\text{homeo}}{\approx} (\mathbb{R}^4, \mathbb{R}^2)$



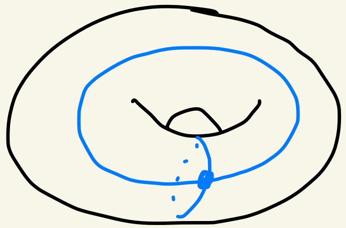
- generically the image of $\Sigma^2 \longrightarrow M^4$ has isolated double point singularities $(2+2=4)$

Why is this an interesting question?

Example:

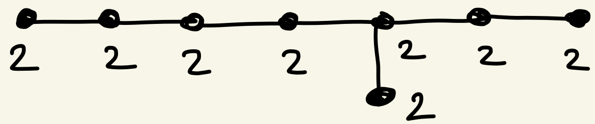
- By Poincaré duality, every closed 4-mfld has an bilinear, unimodular intersection form

$$Q_M : H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) \longrightarrow \mathbb{Z}$$



- e.g. $Q_{S^2 \times S^2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

- $E8 := \begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & 0 & 1 \\ & & & & 1 & 2 & 1 & 0 \\ & & & & 0 & 1 & 2 & 0 \\ & & & & 1 & 0 & 0 & 2 \end{bmatrix}$



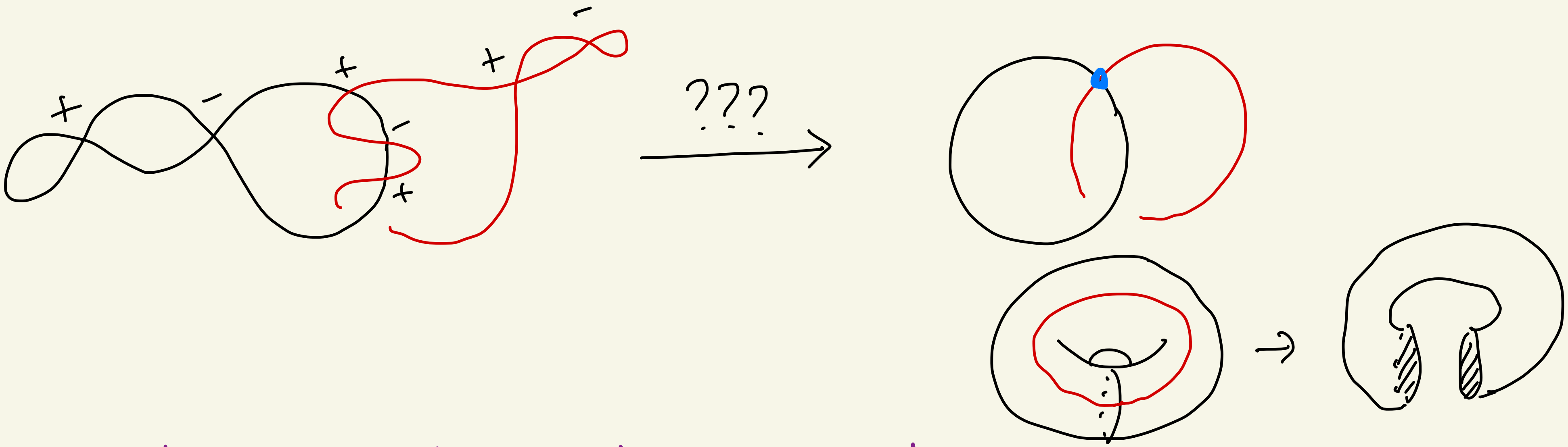
Q: Is $E8 \oplus E8$ the intersection form of a closed, simply connected 4-mfld?

Idea:

The K3 surface $:= \{[x,y,z,w] \in \mathbb{CP}^3 \mid x^4+y^4+z^4+w^4=0\}$

$$\pi_1(K3) = 1 \implies \pi_2(K3) \cong H_2(K3)$$

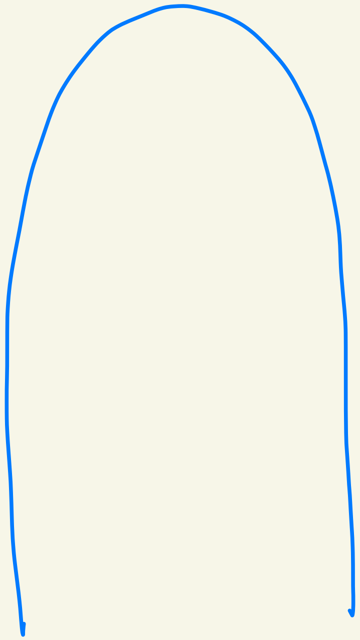
$$Q_{K3} \cong E8 \oplus E8 \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

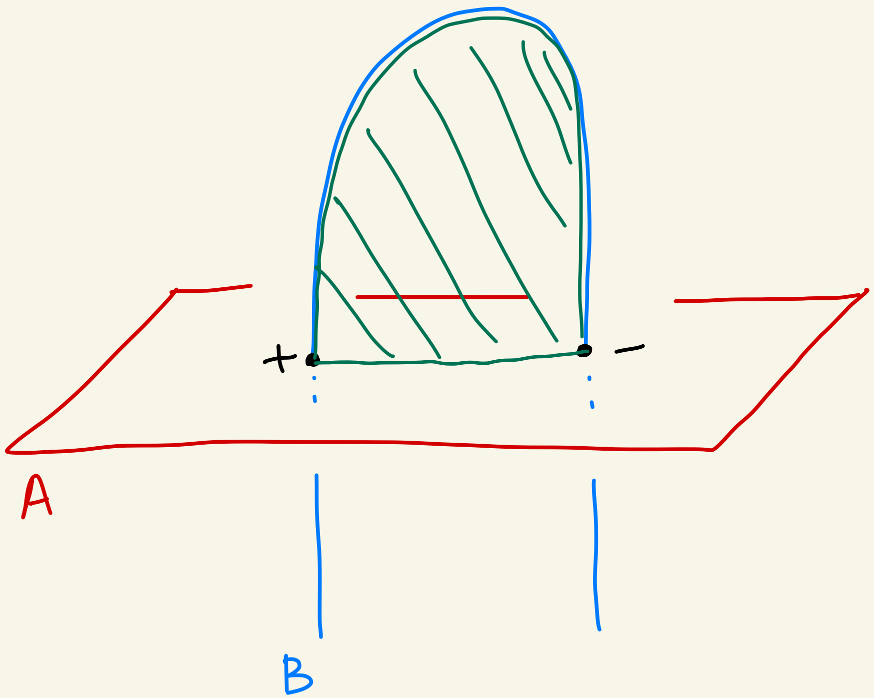


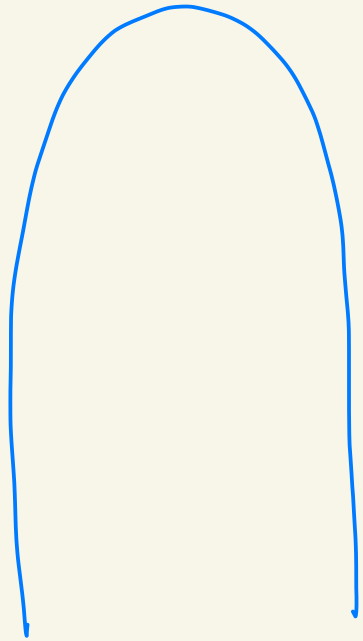
Goal: realise algebra by geometry.

Spoiler: this is possible topologically [Freedman]
but not smoothly [Donaldson]

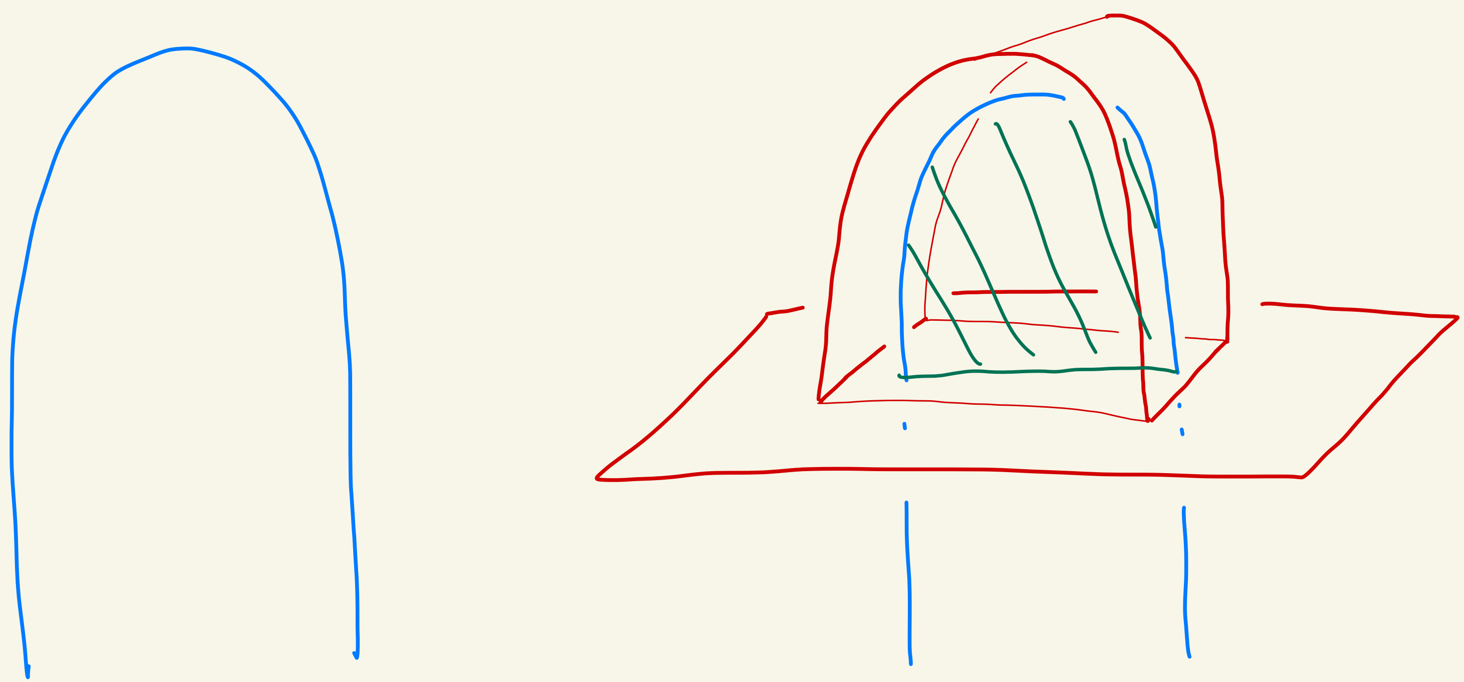
The Whitney trick





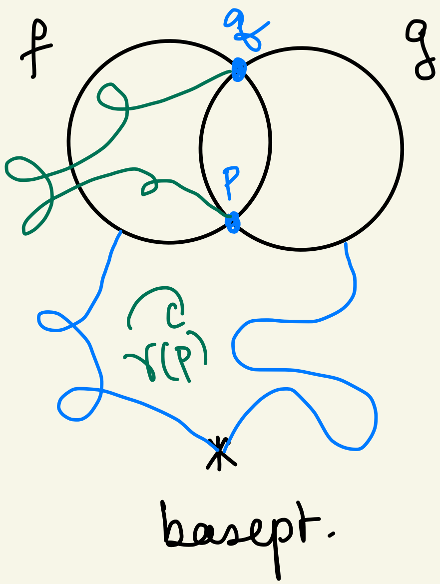


The Whitney trick



- if $\exists$ (framed) embedded Whitney disc, can remove the pair of intersections
- using the Whitney trick, Smale proved the smooth h-cob theorem in dim ≥ 5 $\Rightarrow$ Poincaré conjecture
- what about dimension 4?

Intersection numbers



$$\lambda(f,g) := \sum_{p \in f \pitchfork g} \varepsilon(p)\,\gamma(p) \quad \in \mathbb{Z}[\pi_1 M]$$

well-defined if f, g simply connected [modulo the choice of whiskers]

$\lambda(f,g) = 0 \iff$ all pts $p \in f \pitchfork g$ paired by gen. immersed Wh discs w. framed, embedded, pairwise disjoint boundaries

gen coll. of Wh discs

Self-intersection number $\mu(f) = 0 \iff$ all pts $\in f \pitchfork f$ are paired by gen. coll. of Wh discs

f, g are alg. dual if $\lambda(f,g) = 1 \iff$ all but one pt in $f \pitchfork g$ are paired

f, g are geom. dual if $f \pitchfork g = \{pt\}$

Breakthrough result : Disc embedding theorem (Casson, Freedman'82, Freedman - Quinn'90)

M^4 connected, topological manifold. $\pi_1 M$ good

$\Sigma = \sqcup \Sigma_i$ compact surface, each Σ_i simply connected

$\Sigma_i = D^2$ or S^2

$$\begin{tikzcd} F : \Sigma \arrow[r, looseness=0, "\circ"] & M \\ \partial\Sigma \arrow[u, hook] \arrow[r, hook] & \partial M \arrow[u, hook] \end{tikzcd}$$

generic immersion

$F = \sqcup f_i \qquad f_i : \Sigma_i \looparrowright M$

such that • algebraic intersection numbers of F vanish

$\lambda(f_i, f_j) = \mu(f_i) = 0$

• $\exists\, G : \sqcup S^2 \looparrowright M$ framed alg. dual to F

$G = \sqcup g_i$ trivial norm. bundle. $\lambda(f_i, g_j) = \delta_{ij}$

Then F is (reg.) htpic rel ∂ to a loc. flat emb $\bar{F}$

[with geom dual spheres $\bar{G}$ with $G \simeq \bar{G}$.] $\pi_1 \neq 1$ Powell - R. Teichner '20

Consequences of the disc embedding theorem

- h-cobordism theorem,
  s-cobordism thm (good π_1)
- surgery sequence exact (good π_1)
- Poincaré conjecture

Quinn : annulus thm $\Longrightarrow$ connected sum of TOP 4-mflds well-defined.

Good groups

- abelian gps, finite gps, solvable groups, ...
- gps of subexp growth [Krushkal-Quinn, Freedman-Teichner]
- closed under subgps, quotients, direct limits, extensions.
- open e.g. whether $\mathbb{Z} * \mathbb{Z}$ good

~~Disc~~ Surface embedding theorem (Casson, Freedman '82, Freedman–Quinn '90
Stong, Kasprowski–Powell–R. Teichner '20+

M^4 connected, topological manifold. $\pi_1 M$ good

$\Sigma = \sqcup \Sigma_i$ compact surface, ~~each Σ_i simply connected~~

$$\begin{array}{ccc} F: \Sigma & \looparrowright & M \\ \cup & & \cup \\ \partial\Sigma & \hookrightarrow & \partial M \end{array} \quad \text{generic immersion}$$

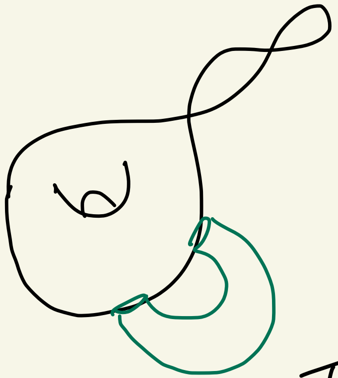
such that
- algebraic intersection numbers of F vanish
- $\exists\, G: \sqcup S^2 \looparrowright M$ ~~framed~~ alg. dual to F

Then F is (reg.) htpic rel ∂ to a loc. flat emb $\bar{F}$ with geom dual spheres $\bar{G}$ with $G \simeq \bar{G}$

iff $km(F) \in \mathbb{Z}/2$ vanishes

Kervaire–Milnor invariant.

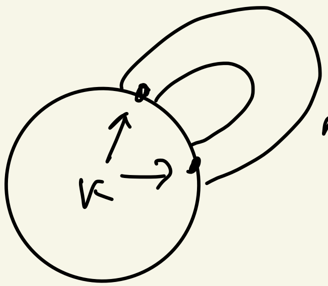
Corollary 1: $F : \Sigma^2 \looparrowright M^4$ with
- Σ connected
- alg int numbers vanish
- $\exists G$ alg dual sphere



$F' := F \cup$ trivial tube

Then f' is (reg) htpic to an embedding

Corollary 2: $F : \Sigma^2 \looparrowright M^4$ with
- Σ connected, $g(\Sigma) > 0$
- alg int numbers vanish
- $\exists G$ alg dual sphere
- $\pi_1 M = 1$



± 1 framed 2-handle along K

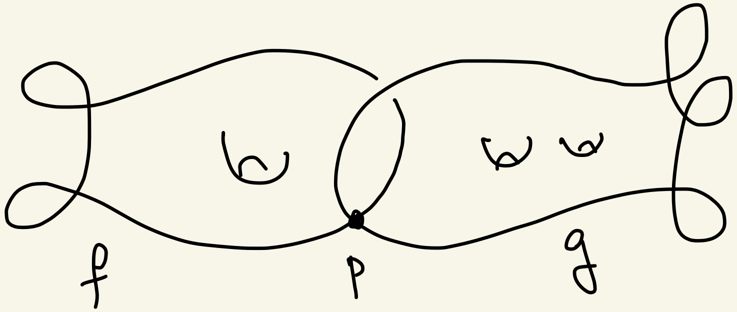
Then f is (reg) htpic to an embedding

Corollary: $g^{TOP}_{sh,\pm 1}(K) \leq 1 \quad \forall K$

[FMNOPR'20]

[In fact, $g^{TOP}_{sh,\pm 1}(K) = \mathrm{Arf}(K) \quad \forall K$]

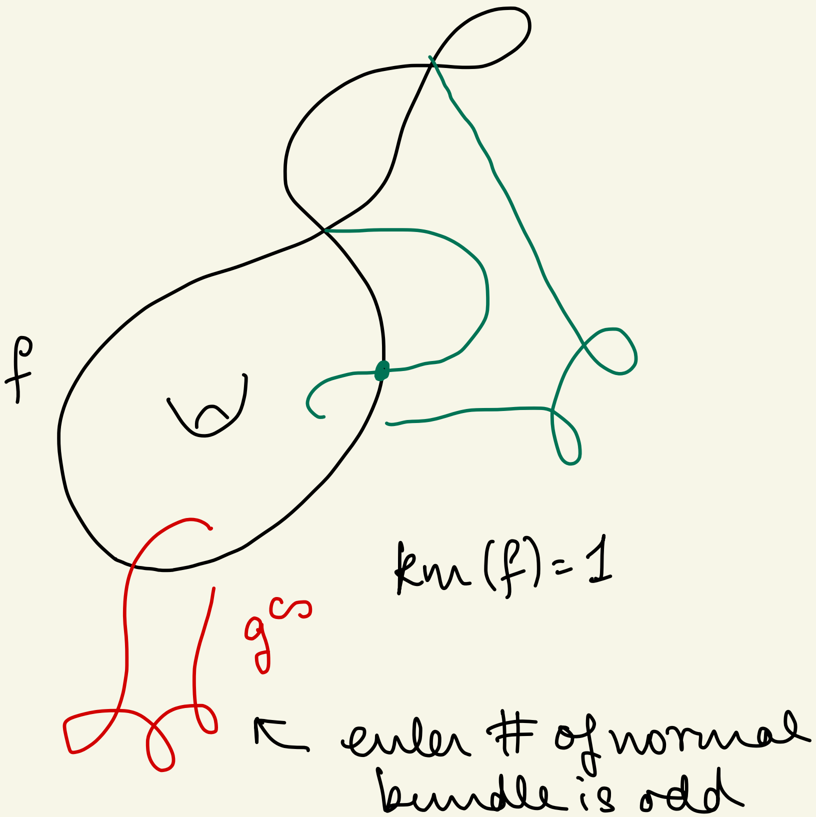
Definition of invariants:



$\lambda(f,g)$ not well defined in $\mathbb{Z}[\pi_1 M]$!

$\lambda(f,g)=0 \iff$ all pts in $f \pitchfork g$ paired by gen imm. coll of wh discs

$$:= \sum_\ell |\operatorname{Int} W_\ell^{cs} \pitchfork F^{cs}| \mod 2$$



In general, $\lambda(F,F) = \mu(F) = 0$

$\implies \exists \{W_\ell\}$ gen coll. of wh discs for $F \pitchfork F$

$F^{cs} \subseteq F$ subsurface w. <u>twisted</u> dual spheres

$\{W_\ell^{cs}\} \subseteq \{W_\ell\}$ pairing ints of F^{cs}

$$\boxed{Km(F;\{W_\ell\}) := \sum_\ell |\operatorname{Int} W_\ell^{cs} \pitchfork F^{cs}| \mod 2}$$

Thanks for your attention!