How to disprove the triangulation conjecture

A TOP space M is <u>triangulated</u> if it is homeo to (the geom real. of) a simplicial complex.

<u>Triangulation conjecture</u> Poincaré 1899, Kneser 1926

Every TOP manifold is <u>triangulable</u>

$n \leq 3$	Yes	Radó (1925) Moise (1952)
$n = 4$	No	e.g. E_8 manifold 1980s
$n \geqslant 5$	No	Manolescu (2013)

$\Theta^n_{\mathbb{Z}} := \{\text{oriented PL } \mathbb{Z}HS^n\} / \text{PL } \mathbb{Z}H \text{ cob.}$

Kervaire (1969) $\Theta^{n\neq 3}_{\mathbb{Z}} = 0$

Rokhlin (1952) $\Theta^3_{\mathbb{Z}} \overset{\mu}{\twoheadrightarrow} \mathbb{Z}/2$ $\quad \mu(\text{Poincaré}) = 1$

Manolescu: $\forall Y$ $\mathbb{Z}HS^3$ with $\mu(Y) = 1$, $2[Y] \neq 0 \in \Theta^3_{\mathbb{Z}}$.

Specifically, defined $\beta : \Theta^3_{\mathbb{Z}} \longrightarrow \mathbb{Z}$

s.t. $\beta(Y) \equiv \mu(Y) \bmod 2$,

& $\beta(-Y) = -\beta(Y)$

Then $2Y = 0 \implies Y = -Y \implies \beta(Y) = \beta(-Y) = -\beta(Y)$

$\implies \beta(Y) = 0$

$\implies \mu(Y) = 0$

[Galewski-Stern 1980, Matumoto 1978]

Let M be a TOP n-mfld with $\begin{cases} n \geq 7 \\ n \geq 6 \text{ if } \partial M \text{ compact} \\ n \geq 5 \text{ if } \partial M = \emptyset \end{cases}$

Any such M can be triangulated

if and only if

$\exists\, Y^3$ $\mathbb{Z}HS^3$ s.t. $\mu(Y) = 1$ and $2[Y] = 0 \in \Theta^3_{\mathbb{Z}}$

A combinatorial/PL triangulation of a TOP space is a triangulation where the link of every vertex is a PL sphere (or ball, if $\partial \neq \emptyset$).

Let M^n be a triangulated closed TOP mfld w. $n \geq 3$. Then the link of every vertex is a simply connected homology sphere

$\exists$ non-combinatorial triangulations of DIFF mflds.

e.g. S^5

$\exists$ TOP mflds without comb. triangulations

e.g. $E8 \times S^k \quad k \geqslant 0$

(in fact these are triangulated, but not comb. triangulated for $k \geqslant 1$)

E8 is not triangulable

E8 does not admit a comb. triang.

e.g. Rokhlin's theorem $\Longrightarrow$ E8 not smooth

$\Downarrow$

E8 not PL

Suppose triangulated.

$v \in$ vertex.

$$Lk(v) \xrightarrow[\text{Pe' conj}]{\text{Option 1}} \text{PL-sphere} \Longrightarrow \text{comb } \Delta \overset{n}{\doteq}. \not\Longleftarrow$$

$\downarrow$ Option 2.

$\lambda(Lk(v)) = 0$ Casson invt $\Rightarrow \mu(Lk(v)) = 0$

$\not\Longleftarrow$

$$\mu(Lk(v)) = \frac{1}{8}\sigma(E8) = 1$$

Warmup: Kirby-Siebenmann invariant

$M^{n\geq 5}$ a TOP mfld closed

$\rightsquigarrow$

$\exists$ TOP tangent bundle i.e. $\mathbb{R}^n$ - fibres with str. gp
$\mathrm{Homeo}_0(\mathbb{R}^n) =: TOP(n)$
$TOP := \lim_{n\to\infty} TOP(n)$

- $\exists$ classifying space BTOP and a map

$$\begin{tikzcd} & TOP/PL \simeq_{KS} K(\mathbb{Z}/2,3) \arrow[d] \\ & BPL \arrow[d] \\ M \arrow[r, "\tau_M"] \arrow[ur, dashed, "\exists\alpha"] & BTOP \arrow[d] \\ & B(TOP/PL) \simeq K(\mathbb{Z}/2,4) \end{tikzcd}$$

lift α exists iff $\exists$ PL str. on $M\times\mathbb{R}^k$ for some k

$\rightsquigarrow$ product str. theorem (KS) ($M^{n\geq 5}_{\text{closed}}$)

M has a PL str.

$ks(M)\in H^4(M;\mathbb{Z}/2)$ is the unique obstr. to the existence of this lift α.

Alternative definition of RS. (Cohen 1970, Sullivan 1969, Martin 1973)

Let M^n be endowed w. a triangulation K
not necessarily comb.

$\exists ?\ N \xrightarrow{f} M \leftarrow H_*$ mfld
PL mfld
fibres acyclic? contractible?

M closed, orientable for simplicity.

Define $c(K) := \sum_{\sigma \in K^{n-4}} [LK(\sigma)]\,\sigma \in H_{n-4}(M; \Theta^3_{\mathbb{Z}})$

lk of simplex not nec. vertex!

$\cong H^4(M; \Theta^3_{\mathbb{Z}})$

$$0 \longrightarrow \mathrm{Ker}\,\mu \longrightarrow \Theta^3_{\mathbb{Z}} \xrightarrow{\mu} \mathbb{Z}/2 \longrightarrow 0$$

$$\cdots \longrightarrow H^4(M; \Theta^3_{\mathbb{Z}}) \xrightarrow{\mu_*} H^4(M; \mathbb{Z}/2) \xrightarrow{\delta} H^5(M; \mathrm{Ker}\,\mu) \longrightarrow \cdots$$

$$c(K) \longmapsto RS(M)$$

Outline of Galewski-Stern proof (1980)

M^n a TOP mfld, assume $n \geq 5$, M closed

$\rightsquigarrow$

TOP tangent bundle $M \xrightarrow{\tau_M} BTOP$

$\exists$ classifying space $BTRI$ and a map $BTRI \to BTOP$

s.t. M admits a triangulation

if and only if

$$\begin{tikzcd} & TOP/TRI \arrow[d] \\ & BTRI \arrow[d] \\ M \arrow[ur, dashed, "?\exists \alpha"] \arrow[r, "\tau_M"] & BTOP \arrow[d] \\ & B(TOP/TRI) \end{tikzcd}$$

[in particular they prove a product str. theorem]

Homotopy type of TOP/TRI

$\exists$ homotopy commutative diagram

$$
\begin{tikzcd}
K(\mathbb{Z}/2,3) \simeq TOP/PL \arrow[dr] & & \\
K(\Theta^3_{\mathbb{Z}},3) \simeq TRI/PL \arrow[r] & BPL \arrow[r] \arrow[dr] & BTRI \arrow[d] \\
 & & BTOP
\end{tikzcd}
$$

Martin

Homotopy exact sequence of the triple $(BTOP, BTRI, BPL)$

$$
\begin{tikzcd}[column sep=small]
 & \pi_i(TOP/PL) \arrow[d, phantom, sloped, "\cong"] & \pi_i(TOP/TRI) \arrow[d, phantom, sloped, "\cong"] & \pi_{i-1}(TRI/PL) \arrow[d, phantom, sloped, "\cong"] \\
\pi_i(BTRI, BPL) \arrow[r] & \pi_i(BTOP, BPL) \arrow[r] & \pi_i(BTOP, BTRI) \arrow[r] & \pi_{i-1}(BTRI, BPL)
\end{tikzcd}
$$

$$i \neq 3,4, \quad \cancelto{0}{\pi_i(TOP/PL)} \longrightarrow \pi_i(TOP/TRI) \longrightarrow \pi_{i-1}\cancelto{0}{(TRI/PL)}$$

o/w.

$$
\begin{tikzcd}
0 \arrow[r] & \pi_4(TOP/TRI) \arrow[r] & \pi_3(TRI/PL) \arrow[r] & \pi_3(TOP/PL) \arrow[r] \arrow[d, "d", "\cong"'] & \pi_3(TOP/TRI) \arrow[r] & 0 \\
 & & \Theta^3_{\mathbb{Z}} \arrow[u, "e", "\cong"'] \arrow[r, "\mu"] & \mathbb{Z}/2 & &
\end{tikzcd}
$$

$$\Longrightarrow \; TOP/TRI \simeq K(\ker \mu, 4).$$

So far, M TOP mfld, M triangulated iff $\exists$ lift

$$\begin{tikzcd}
 & TOP/TRI \simeq K(\ker\mu, 4) \arrow[d] \\
 & BTRI \arrow[d] \\
M \arrow[r, "\tau_M"] \arrow[ur, dashed, "\exists?\alpha"] & BTOP \arrow[d] \\
 & B(TOP/TRI) \simeq K(\ker\mu, 5)
\end{tikzcd}$$

Define the triangulation obstruction $\nabla(M) \in H^5(M; \ker\mu)$, the unique obstruction to finding such a lift

How big is $\mathrm{Ker}(\mu : \Theta^3_{\mathbb{Z}} \longrightarrow \mathbb{Z}/2)$?

Fintushel-Stern 1985, 1990 : $\exists\, \mathbb{Z}^\infty < \Theta^3_{\mathbb{Z}}$
Furuta 1990

gen by $\Sigma(2,3,6i-1)$, $i \geq 1$

Frøyshov 2002 : $\exists\, \mathbb{Z}$ summand of $\Theta^3_{\mathbb{Z}}$

gen by $P := \Sigma(2,3,5)$ Poincaré sphere

Dai-Hom-Stoffregen-Truong 2019 : $\exists\, \mathbb{Z}^\infty$ summand of $\Theta^3_{\mathbb{Z}}$

gen by $\Sigma(2i+1, 4i+1, 4i+3)$, $i > 1$

Open questions : $\exists$? torsion in $\Theta^3_{\mathbb{Z}}$?

Is $\Theta^3_{\mathbb{Z}} \cong \mathbb{Z}^\infty$?

$$0 \to \operatorname{Ker}\mu \longrightarrow \Theta^3_{\mathbb{Z}} \xrightarrow{\mu} \mathbb{Z}/2 \longrightarrow 0$$

$$\cdots \longrightarrow H^4(M;\Theta^3_{\mathbb{Z}}) \xrightarrow{\mu_*} H^4(M;\mathbb{Z}/2) \xrightarrow{\delta} H^5(M;\operatorname{Ker}\mu) \to \cdots$$

$$ks(M) \longmapsto \nabla(M)$$

$$\begin{tikzcd} BPL \arrow[r] \arrow[dr] & BTRI \arrow[d] \\ & BTOP \end{tikzcd}$$

Check out connection to Sullivan's Hauptvermutung [e.g. Ranicki intro to Hauptvermutung book]

The obstruction to finding a section of

$$K(\operatorname{Ker}\mu, 4) \simeq TOP/TRI \longrightarrow BTRI \longrightarrow BTOP$$

is the universal triangulation obstruction $\nabla \in H^5(BTOP;\operatorname{Ker}\mu)$

We have $\nabla(M) = \tau_M^*(\nabla)$

and $\delta(ks) = \nabla$ where ks = univ PLing ob.

Suppose we have a splitting

i.e. $0 \to \underset{\substack{\| \\ \mathrm{Ker}\,\mu}}{\pi_4(TOP/TRI)} \longrightarrow \Theta^3_{\mathbb{Z}} \underset{\exists r}{\overset{\mu}{\rightleftarrows}} \mathbb{Z}/2 \longrightarrow 0$ $\quad \mu \circ r = id$

Find Y $\mathbb{Z}HS^3$ $\mu(Y) = 1$
$2Y = 0 \in \Theta^3_{\mathbb{Z}}$

$$\cdots \longrightarrow H^4(BTOP; \Theta^3_{\mathbb{Z}}) \underset{r_*}{\overset{\mu_*}{\rightleftarrows}} H^4(BTOP; \mathbb{Z}/2) \xrightarrow{\delta} H^5(BTOP; \mathrm{Ker}\,\mu) \longrightarrow \cdots$$

$$ks \overset{0}{\longmapsto} \nabla$$

$id = \mu_* \circ r_* \Rightarrow \mu_*$ surj $\Rightarrow \delta$ 0-map $\Rightarrow \nabla = 0 \Rightarrow \nabla(M) = 0$ $\forall M$.

What about necessity?

$\exists$ closed TOP M^5 with $Sq^1(ks(M)) \neq 0$ [Galewski-Stern]

where recall Sq^1 is the Bockstein homom$^{\underline{m}}$

associated to $0 \to \mathbb{Z}/2 \xrightarrow{\cdot 2} \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0$

i.e. $Sq^1 : H^4(M;\mathbb{Z}/2) \longrightarrow H^5(M;\mathbb{Z}/2)$

$W := *(CP^2 \# \overline{CP}^2)$ $\quad f: W \circlearrowleft \cong$ o.r. $\quad M := W \times I / (w,0) \sim (f(w),1)$

$\cong$

$CP^2 \# *\overline{CP}^2$

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \sim -\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Manolescu lecture notes
(attributed to Kronheimer)

Suppose M has a triangulation K.

Let $\Theta :=$ subgp of $\Theta^3_{\mathbb{Z}}$ gen by 3-dim links of K.

$i : \Theta \hookrightarrow \Theta^3_{\mathbb{Z}}$ inclusion.

Suppose every $[Y] \in \Theta$ with $\mu(Y) = 1$ does not have order 2 in $\Theta^3_{\mathbb{Z}}$

Goal: define η

$$\begin{tikzcd}
 & & \Theta \arrow[r, "i"] \arrow[d, "\eta"'] & \Theta^3_{\mathbb{Z}} \arrow[d, "\mu"] & \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "\cdot 2"] & \mathbb{Z}/4 \arrow[r, "r"] & \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

Goal: define η

$$\begin{tikzcd} & & \Theta \arrow[r, "i"] \arrow[d, "\eta"'] & \Theta^3_{\mathbb{Z}} \arrow[d, "\mu"] & \\ 0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "\cdot 2"] & \mathbb{Z}/4 \arrow[r, "r"] & \mathbb{Z}/2 \arrow[r] & 0 \end{tikzcd}$$

$\Theta = \langle Y_1 \rangle \oplus \cdots \oplus \langle Y_r \rangle$ for some $Y_i \in \Theta^3_{\mathbb{Z}}$

with $\langle Y_i \rangle$ the cyclic subgp gen. by Y_i.

- if $\mu(Y_i) = 0$, define $\eta(Y_i) = 0$
- if $\mu(Y_i) = 1$ and $\langle Y_i \rangle \cong \mathbb{Z}$, define η as reduction mod 4
- if $\mu(Y_i) = 1$ and $\langle Y_i \rangle \cong \mathbb{Z}/p^k$ for some prime p

note $p^k \neq 2$ by hypothesis

p^k = even since $\mu(Y_i) = 1 \Rightarrow p^k = 4q$ for some q

Define η as reduction mod 4

$$\begin{tikzcd}
 & & \Theta \arrow[r, "i"] \arrow[d, "r"'] & \Theta^3_{\mathbb{Z}} \arrow[d, "\mu"] & \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "\cdot 2"] & \mathbb{Z}/4 \arrow[r, "r"] & \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

Recall $c(K)$ from before.

$\tilde{c}(K) \in H^4(M;\Theta)$ with $\quad i_*\tilde{c}(K) = c(K) \in H^4(M;\Theta^3_{\mathbb{Z}})$

$$H^4(M;\Theta^3_{\mathbb{Z}}) \xrightarrow{\mu_*} H^4(M;\mathbb{Z}/2)$$
$$c(K) \longmapsto ks(M)$$

Then

$$0 \neq Sq^1(ks(M)) = Sq^1(\mu_* c(K)) = Sq^1(\mu_* i_* \tilde{c}(K)) = Sq^1(r_* r_* \tilde{c}(K))$$
$$= 0 \quad [Sq^1 r_* = 0]$$

$\Rightarrow\!\Leftarrow$

Universal non triangulable manifolds

Galewski - Stern:

If $\exists\, M^n$ TOP closed triangulated mfld w. $n \geq 5$, $Sq^1(ks(M)) \neq 0$

then all TOP closed mflds w. $n \geq 5$ can be triangulated.

$\exists$ N not Δ'd $\Longrightarrow$ SES does not split $\Longrightarrow$ $\left(\begin{array}{l} \forall\, \mu(Y) = 1 \\ \Longrightarrow \text{order } Y \neq 2 \end{array} \right)$

$\Downarrow$

define γ $\Longrightarrow\!\!\!\!\!\!/\ \ $

Post - Manolescu : Every $M^{n \geq 5}$ TOP closed mfld w. $Sq^1(ks(M)) \neq 0$ is non triangulable.

Summary: M^n TOP closed mfld, $n \geq 5$

- $ks(M) \in H^4(M;\mathbb{Z}/2)$ is the complete obstruction to the existence of a comb triang. of M.

- $\nabla(M) \in H^5(M;\ker\mu)$ is the complete obstruction to the existence of a triang. of M.

- $\delta(ks(M)) = \nabla(M)$ where $H^4(M;\mathbb{Z}/2) \xrightarrow{\delta} H^5(M;\ker\mu)$

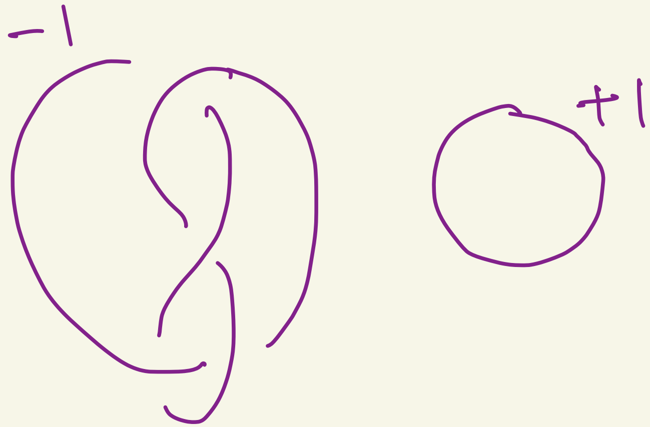
- Every M can be triangulated iff $0 \to \ker\mu \to \Theta^3_{\mathbb{Z}} \to \mathbb{Z}/2 \xrightarrow{\mu} 0$ split

- Manolescu 2013: The sequence does not split.

Notes from post talk discussion

$*CP^2 \# \overline{CP^2}$

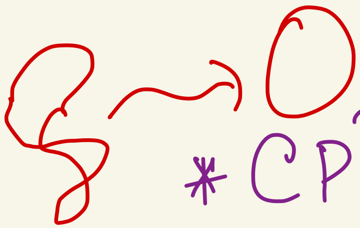
reflect



$*\overline{CP^2} \# CP^2$

$\cong$

$\overline{CP^2} \# *CP^2$



$*CP^2$

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$\overline{CP^2}$

$*CP^2$

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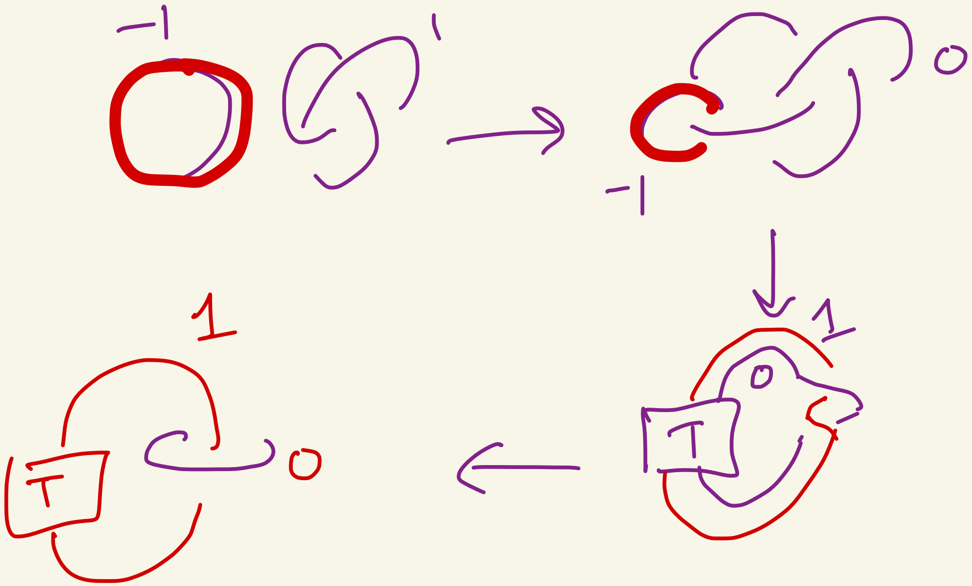
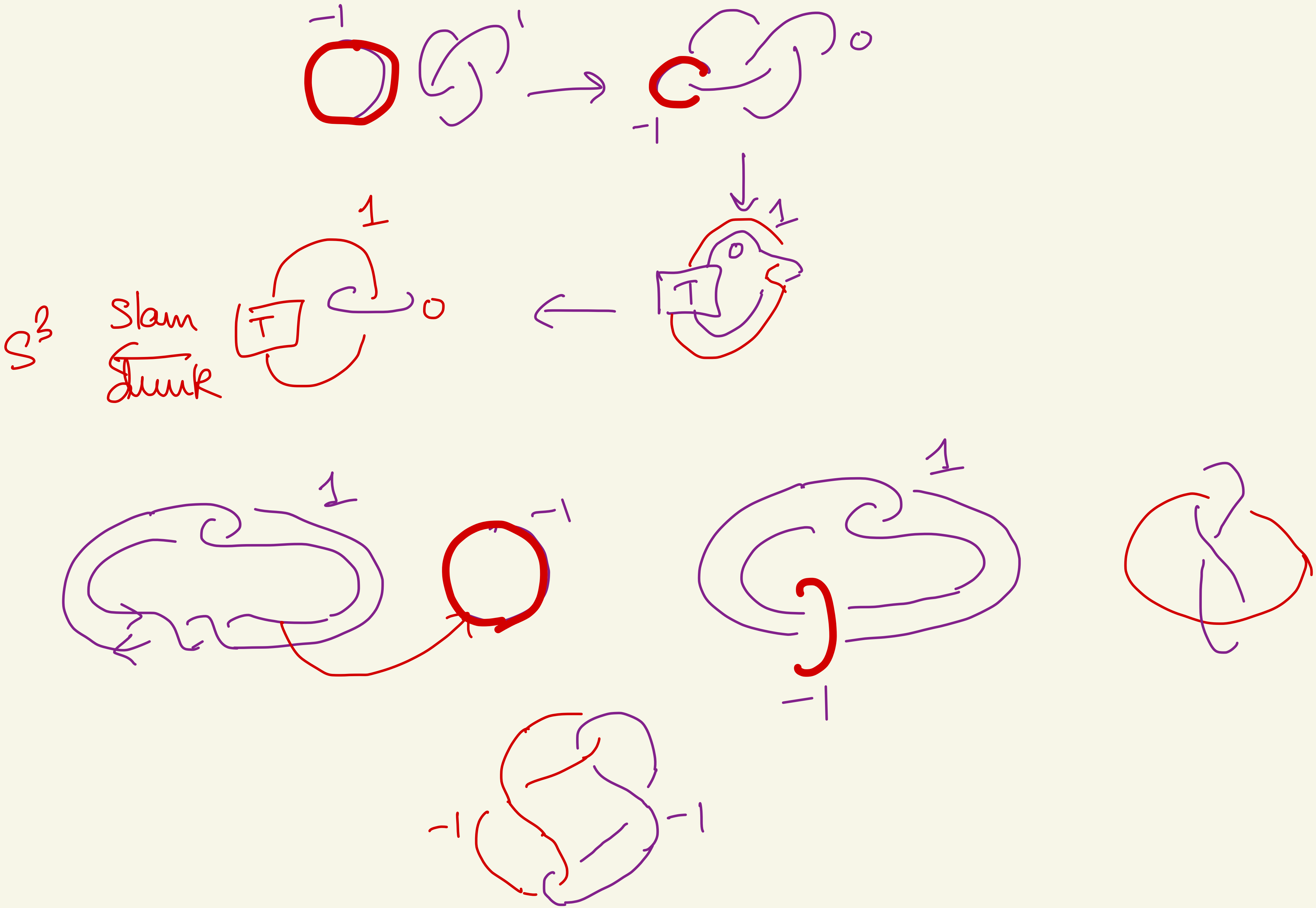
$\overline{CP^2}$



S^3 Slam dunk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