UNC AWM lecture series
Oct 26, 2020

Embedding surfaces in 4-manifolds

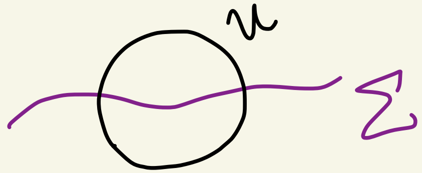
joint with
Daniel Kasprowski
Mark Powell
Peter Teichner

Embedding surfaces in 4-manifolds

(joint w. Kasprowski, Powell, Teichner)

Q: Given a map of a surface in a 4-mfld, when is it homotopic to a (loc. flat or smooth) embedding?

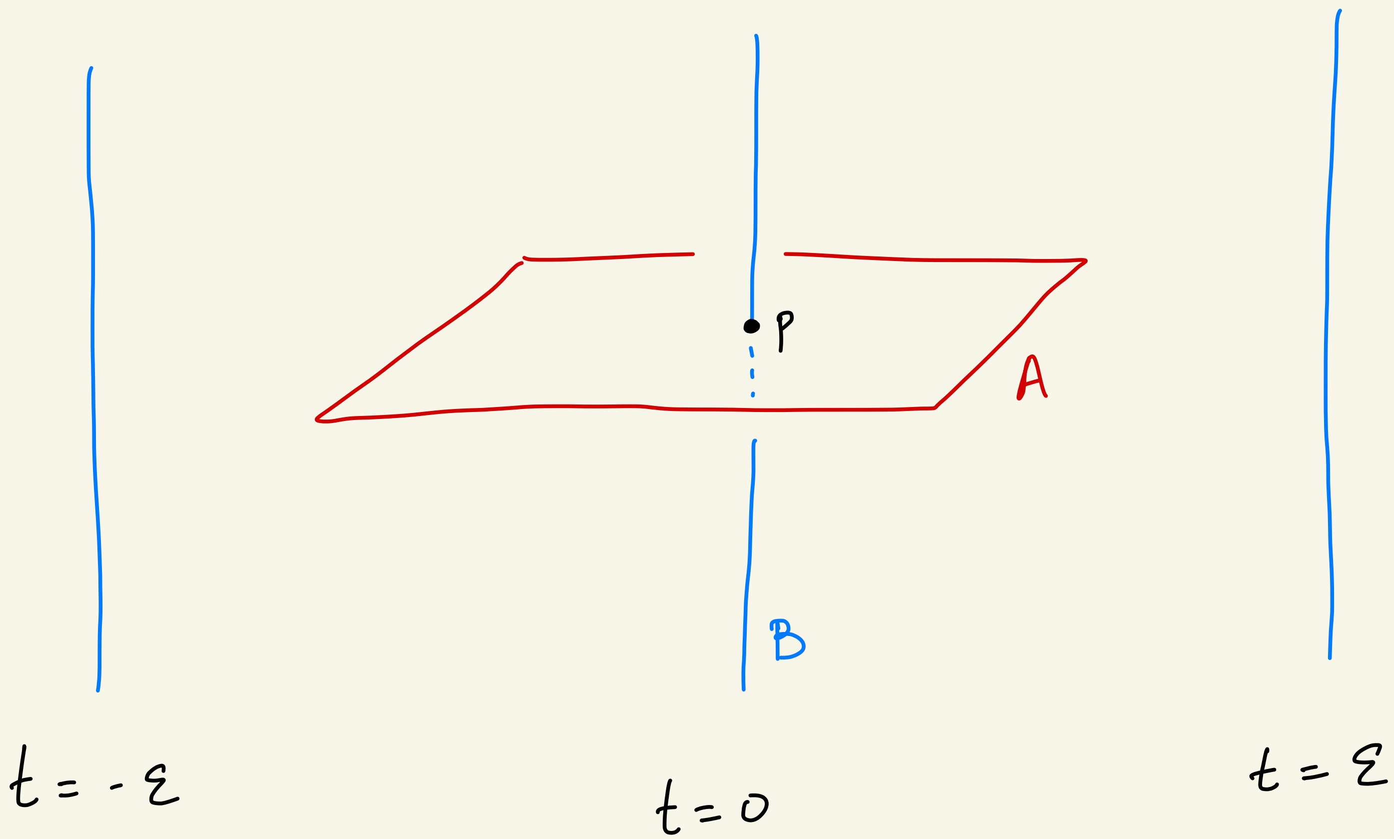
- an embedding $\Sigma \subset M$ is loc. flat if each pt in Σ has a nbd $\mathcal{U}$ s.t. $(\mathcal{U}, \mathcal{U} \cap \Sigma) \underset{\text{homeo}}{\approx} (\mathbb{R}^4, \mathbb{R}^2)$



- generically the image of $\Sigma^2 \longrightarrow M^4$ has isolated double point singularities $(2+2=4)$.

Surfaces intersecting in 4-space

$A \pitchfork B = pt$



- if A, B, M oriented, then each $p \in A \pitchfork B$ has a sign.

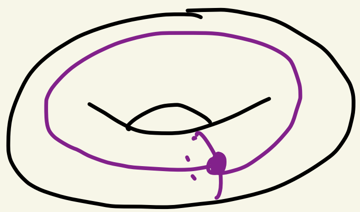
Why is this an interesting question?

Example:

- By Poincaré duality, every closed 4-mfld has an bilinear, unimodular intersection form

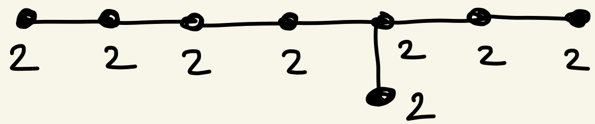
$$Q_M : H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

- e.g. $Q_{S^2 \times S^2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



$S^1 \times S^1$

- $E8 := \begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & 0 & 1 \\ & & & & 1 & 2 & 1 & 0 \\ & & & & 0 & 1 & 2 & 0 \\ & & & & 1 & 0 & 0 & 2 \end{bmatrix}$



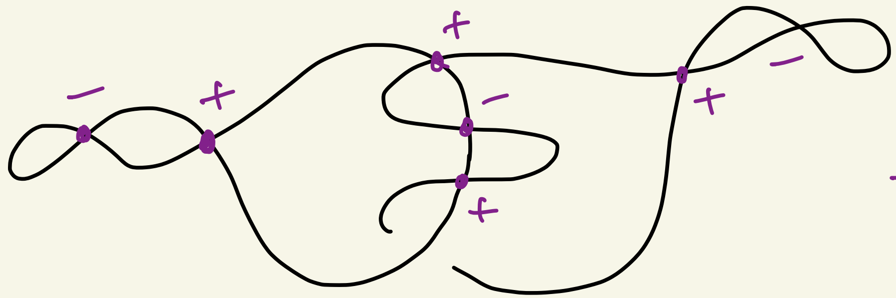
Q: Is $E8 \oplus E8$ the intersection form of a closed, simply connected 4-mfld?

Idea:

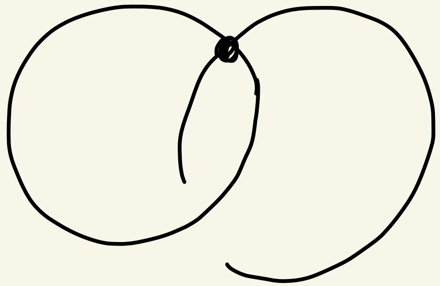
The K3 surface $:= \{[x,y,z,w] \in \mathbb{CP}^3 \mid x^4+y^4+z^4+w^4=0\}$

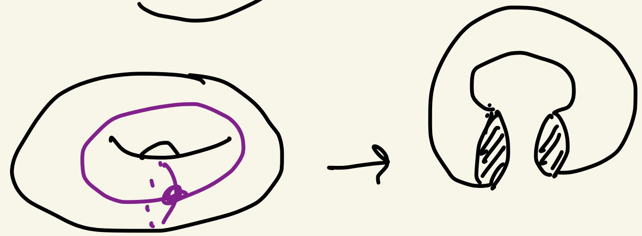
$$\pi_1(K3) = 1 \implies \pi_2(K3) \cong H_2(K3)$$

$$Q_{K3} \cong E8 \oplus E8 \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



want

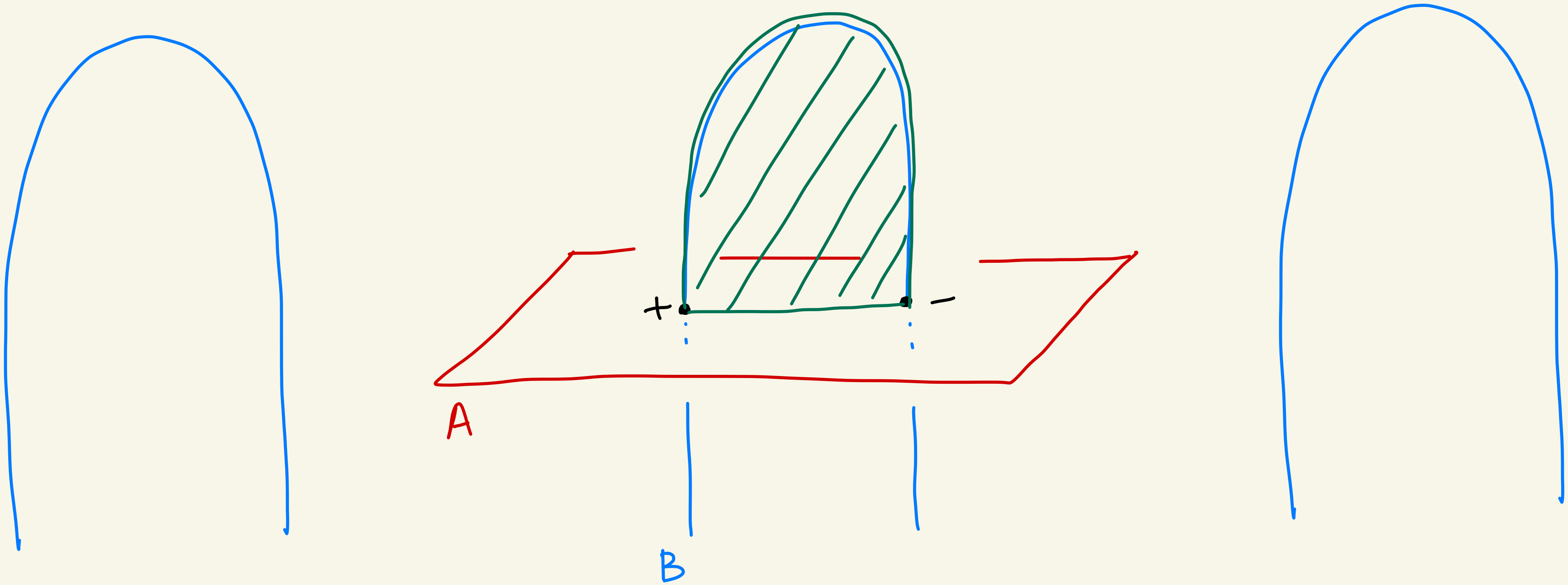




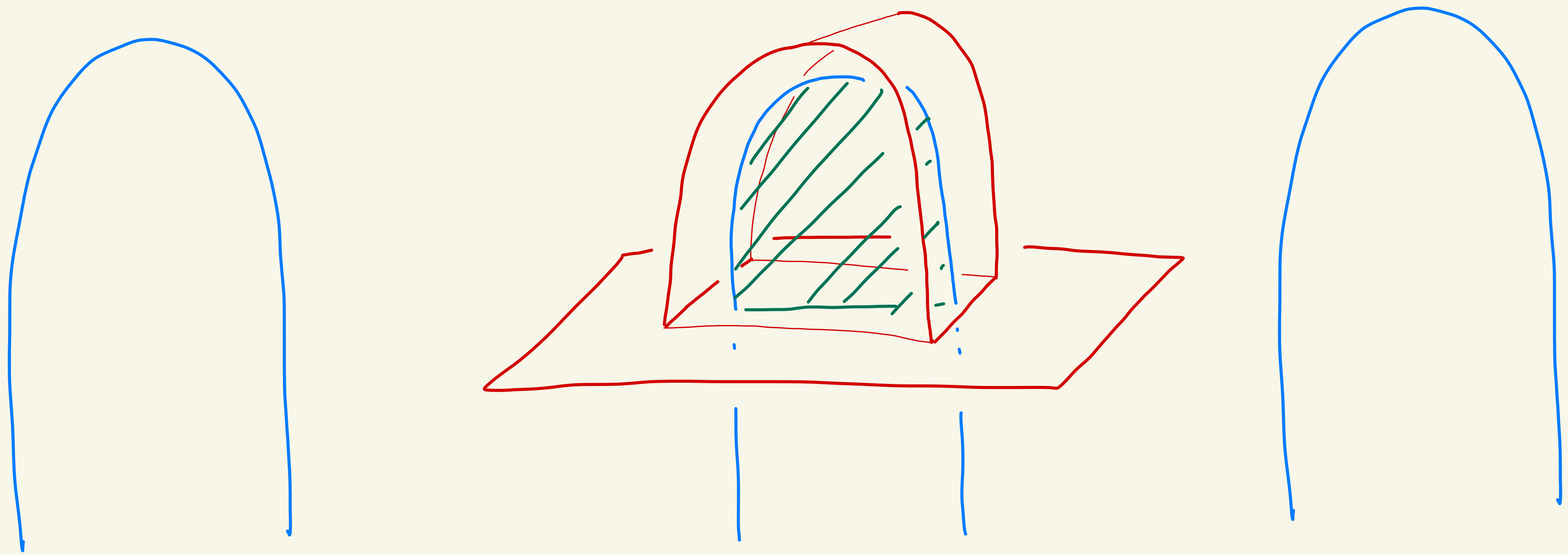
Goal: realise algebra by geometry.

Spoiler: this procedure works in the TOP category! [Freedman Donaldson] does not work in the smooth category.

The Whitney trick

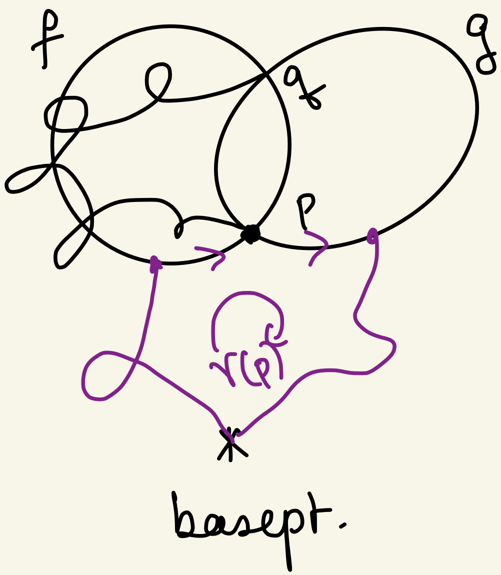


The Whitney trick



- if $\exists$ (framed) embedded Whitney disc, can remove the pair of intersections
- using the Whitney trick, Smale proved the h-cob theorem in dim $\geqslant 5$ $\Rightarrow$ Poincaré conjecture, etc.
- what about dimension 4?

Intersection "numbers"



$$\lambda(f,g) := \sum_{p \in f \pitchfork g} \varepsilon(p)\,\gamma(p) \in \mathbb{Z}[\pi_1 M]$$

if f, g simply connected, this is well defined.
(but depends on the choice of whisker!)

$\lambda(f,g) = 0 \iff$ all pts in $f \pitchfork g$ are paired by immersed Wh. discs with (framed), embedded, disjoint boundaries.

generic coll. of Wh discs := {immersed Wh. discs with (framed), embedded, disjoint boundaries.

Self-intersection number $\mu(f) = 0 \iff$ all pts in $f \pitchfork f$ are paired by gen. coll of Wh discs.

f, g are alg. dual if $\lambda(f,g) = 1 \iff$ all pts in $f \pitchfork g$ <u>except one</u> are paired by gen. coll. of Wh discs.

f, g are geom. dual if $f \pitchfork g = \{pt\}$

Breakthrough result: Disc embedding theorem (Casson, Freedman '82, Freedman-Quinn '90)

M^4 connected, topological manifold. $\pi_1 M$ good

$\Sigma = \sqcup \Sigma_i$ compact surface, each Σ_i simply connected
i.e. $\Sigma_i = D^2$ or S^2

$$\begin{tikzcd} F: \Sigma \arrow[r, squiggly] & M \\ \partial\Sigma \arrow[u, hook] \arrow[r, hook] & \partial M \arrow[u, hook] \end{tikzcd}$$

generic immersion

$F := \{f_i : \Sigma_i \to M\}$

such that
- algebraic intersection numbers of F vanish
  i.e. $\lambda(f_i, f_j) = \mu(f_i) = 0$
- $\exists\, G : \sqcup S^2 \rightsquigarrow M$ framed alg. dual to F
  $G = \{g_i : S^2 \to M\}$ $\lambda(f_i, g_j) = \delta_{ij}$
  framed = normal bundle is trivial

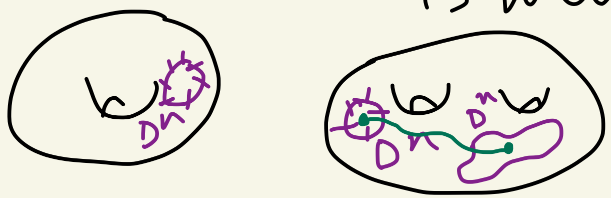
Then F is (reg.) htpic rel ∂ to a loc. flat emb $\bar{F}$

[with geom dual spheres $\bar{G}$ with $G \simeq \bar{G}$] $\pi_1 \neq 1$ Powell-R-Teichner '20

Consequences of the disc embedding theorem

- top h-cobordism in dim 4, (simply connected)
  s-cobordism thm (good gps)
- Poincaré conjecture.
- surgery sequence exact in dim 4 for good gps.

Quinn: annulus thm ⇒ connected sum of top 4-mflds is well defined.



Good groups

- abelian gps, finite gps, solvable groups, ...
- gps of subexp growth [Krushkal-Quinn, Freedman-Teichner]
- closed under subgps, quotients, direct limits, extensions.
- not known whether e.g. $\mathbb{Z} * \mathbb{Z}$ is good.

~~Disc~~ Surface embedding theorem (Freedman '82, Freedman–Quinn '90)
Stong, Kasprowski – Powell – R. – Teichner '20+)

M^4 connected, topological manifold. $\pi_1 M$ good

$\Sigma = \sqcup \Sigma_i$ compact surface, ~~each Σ_i simply connected~~

$$\begin{array}{ccc} F: \Sigma & \looparrowright & M \\ \cup & & \cup \\ \partial\Sigma & \hookrightarrow & \partial M \end{array}$$

generic immersion

such that • algebraic intersection numbers of F vanish

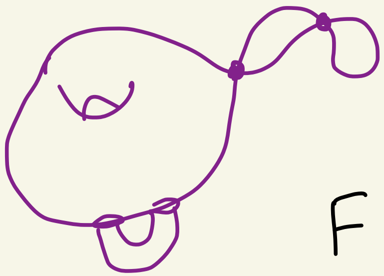
• $\exists\, G: \sqcup S^2 \looparrowright M$ ~~framed~~ alg. dual to F

Then F is (reg.) htpic rel ∂ to a loc. flat emb $\bar{F}$ with geom dual spheres $\bar{G}$ with $G \simeq \bar{G}$

iff $\mathrm{Km}(F) \in \mathbb{Z}/2$ vanishes

Corollary 1: $F : \Sigma^2 \looparrowright M^4$ with

- Σ connected
- alg int numbers vanish
- $\exists G$ alg dual sphere



$F' :=$ result of adding a trivial tube to F

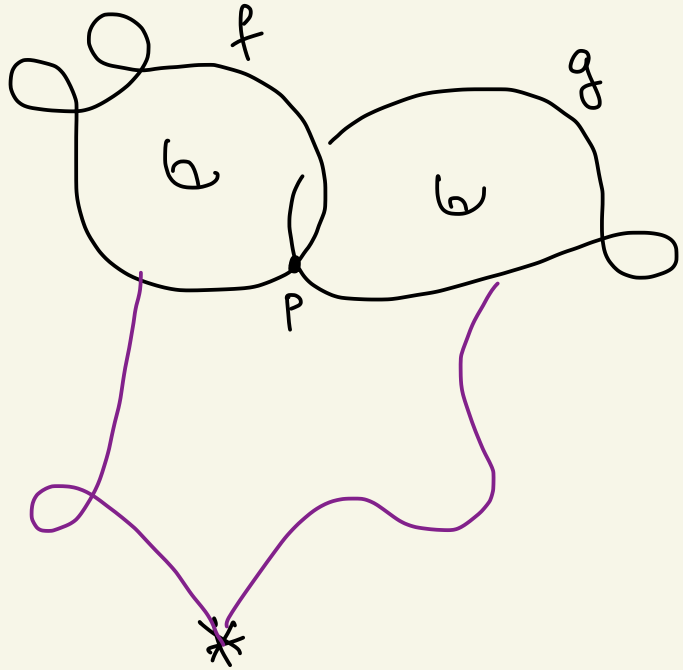
Then F' is (reg) htpic to an embedding

Corollary 2: $F : \Sigma^2 \looparrowright M^4$ with

- Σ connected, $g(\Sigma) > 0$
- alg int numbers vanish
- $\exists G$ alg dual sphere
- $\pi_1 M = 1$

Then F is (reg) htpic to an embedding

Intersection numbers



$\lambda(f,g)$ not well defined in $\mathbb{Z}[\pi_1 M]$!

consider a double coset space.

" $\pi_1(f) \backslash \mathbb{Z}[\pi_1 M] / \pi_1(g)$ "

$\lambda(f,g) = 0 \iff$ all pts in $f \pitchfork g$ paired by gen imm. coll of wh discs

$\mu(f) = 0 \iff$ all ints in $f \pitchfork f$ paired by gen imm coll of wh discs

Definition of the Kervaire-Milnor invariant

- for discs/spheres due to FQ90 §10 + Stong

$\Sigma = \sqcup \Sigma_i$

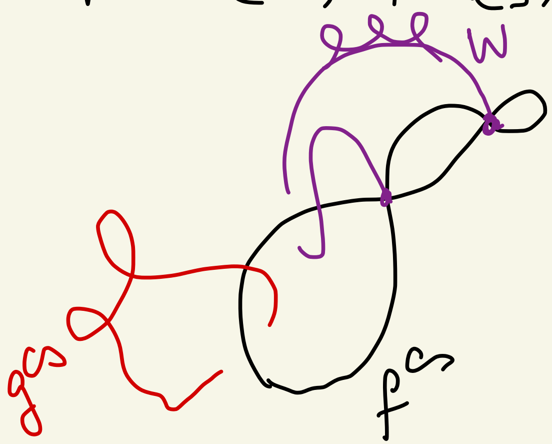
$F : \Sigma \looparrowright M$, alg int numbers of F vanish, $\exists G : \sqcup S^2 \looparrowright M$ alg dual

$\Rightarrow$ ints of F are paired by $\{W_e\}$ gen. imm. coll. of wh discs.

Let $F^{\infty} \subseteq F$ subset with twisted dual spheres
i.e. euler number of normal bundles are odd

Let $\{W_e^{\infty}\} \subseteq \{W_e\}$ subset pairing ints of F^{∞}.

$$km(F; \{W_e\}) := \sum_e | \operatorname{Int} W_e^{\infty} \pitchfork F^{\infty} | \mod 2$$



When is $km(F; \{W_e\})$ independent of $\{W_e\}$?

Proposition (KPRT): $km(F; \{W_e\})$ is well defined

iff

F is b-characteristic

e.g. F is <u>not</u> b-char. if

- $\exists\, S^2 \hookrightarrow M$ with $S \cdot S \not\equiv F \cdot S \mod 2$,
- $\exists\, RP^2 \hookrightarrow M$ with $R \cdot R \not\equiv F \cdot R \mod 2$.

Definition: $$Km(F) = \begin{cases} 0 & \text{if } F \text{ not b-char} \\ km(F; \{W_e\}) & \text{if } F \text{ b-char. for any choice of } \{W_e\} \end{cases}$$

Proof outline: Suppose $\exists \{W_e\}$ s.t. $\mathrm{km}(F;\{W_e\}) = 0 \in \mathbb{Z}/2$

Step 1: By neg htpy, make F and G geom dual (still immersed)

Step 2: Upgrade $\{W_e\}$ and F by neg htpy s.t. $\{\mathrm{Int}\, W_e\} \cap F = \emptyset$

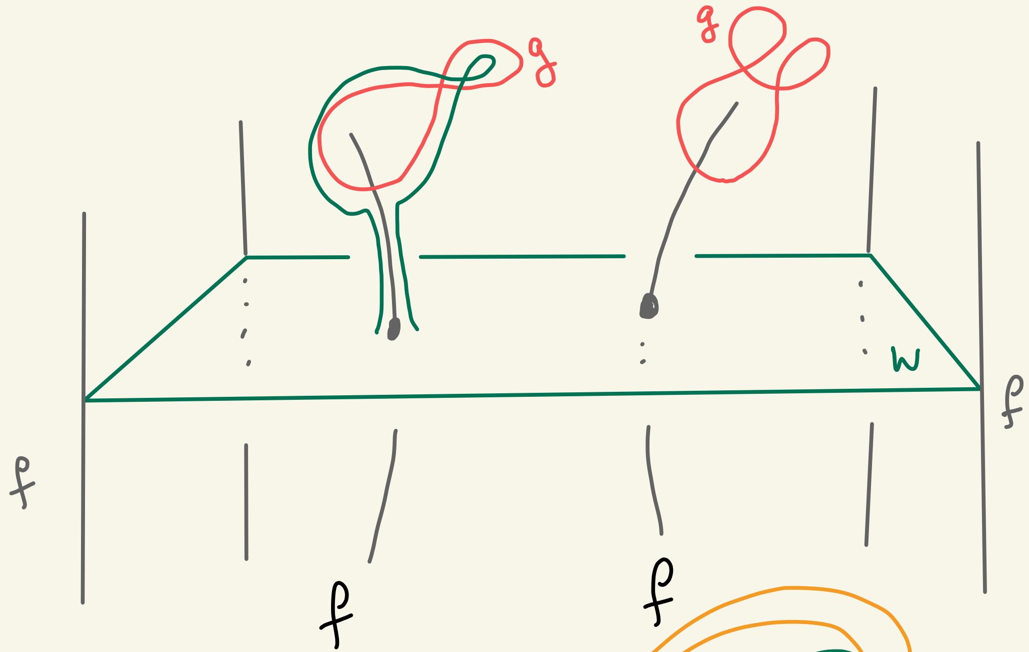
Step 3: Use (Whitney) disc embedding theorem to replace $\{W_e\}$ by $\{V_e\}$ with

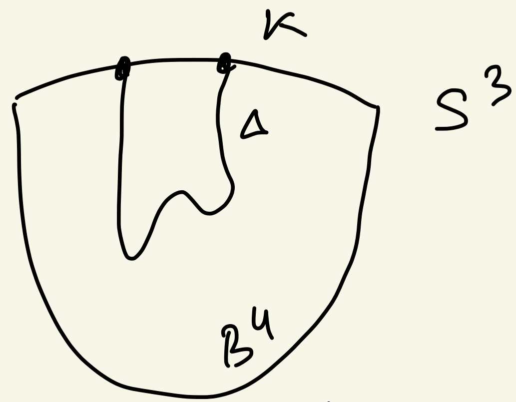
- $\{\mathrm{Int}\, V_e\} \cap F = \emptyset$
- $\{V_e\}$ flat, embedded, disjoint
- geom dual spheres $\{V_e^T\}$ in $M \setminus F$

Step 4: Tube G into $\{V_e^T\}$ to get $\bar{G}$ geom dual to F, disjoint from $\{V_e\}$ (ignore if only care about $\bar{F}$)

Step 5: Wh move F over $\{V_e\}$ to produce desired $\bar{F}$.

Step 2: Upgrade $\{W_e\}$ and F by neg htpy s.t. $\{\text{Int } W_e\} \cap F = \emptyset$

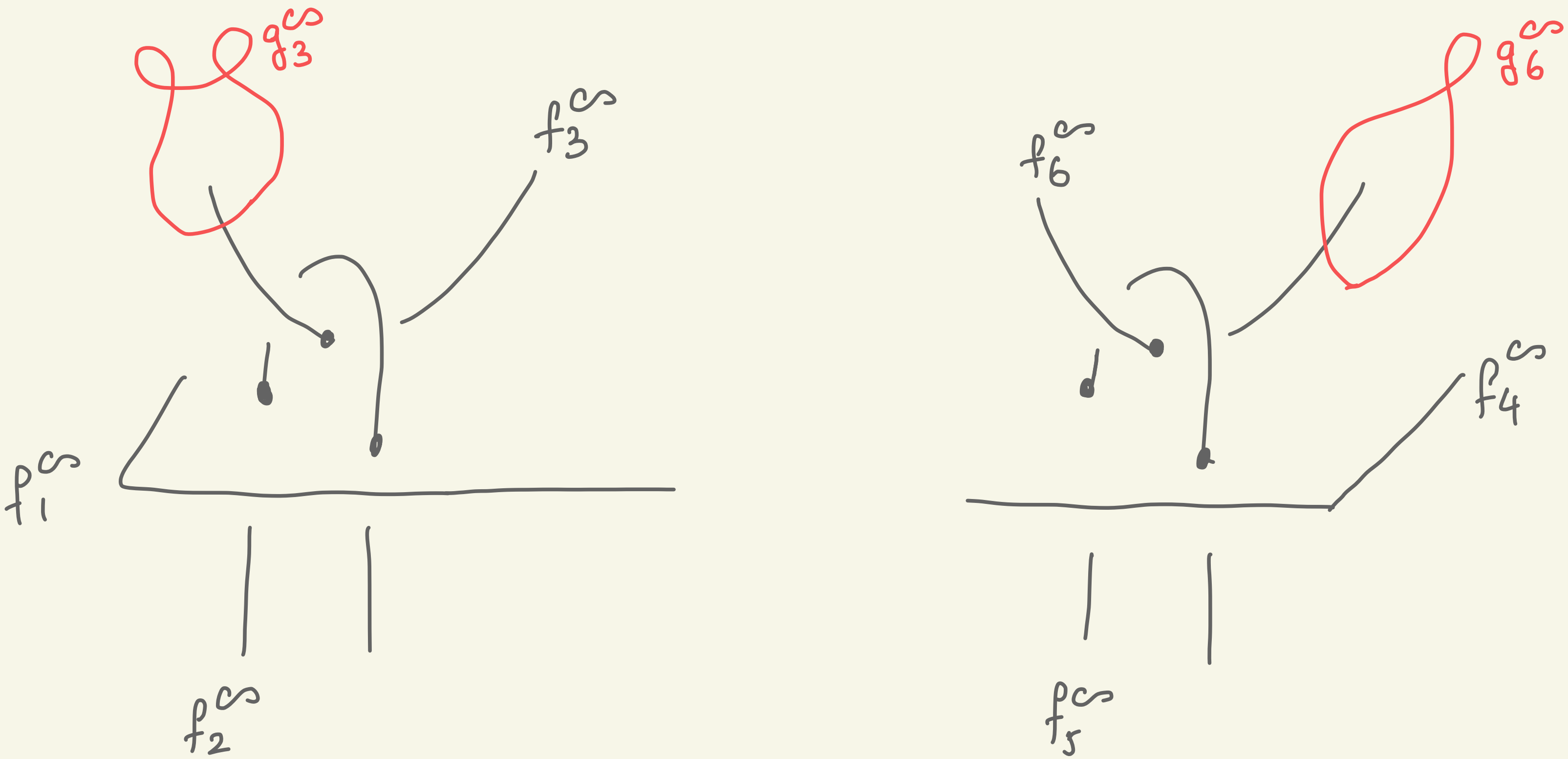




K slice !!

Step 2: Upgrade $\{W_e\}$ and F by neg htpy s.t. $\{\mathrm{Int}\, W_e\} \cap F = \emptyset$

Remaining problem:



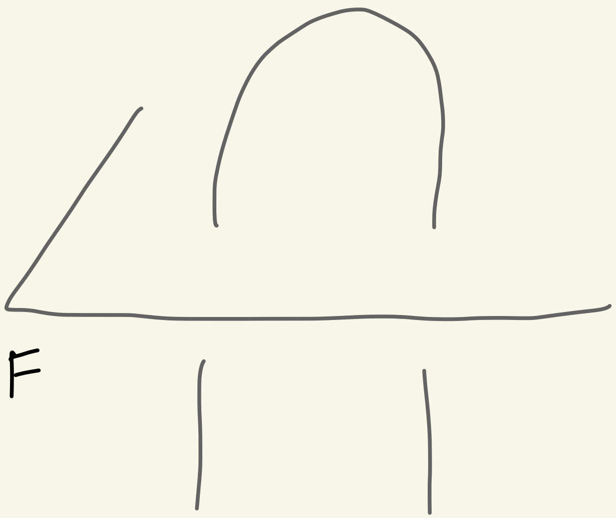
Thanks for your attention!

When is $km(F; \{W_e\})$ independent of $\{W_e\}$?

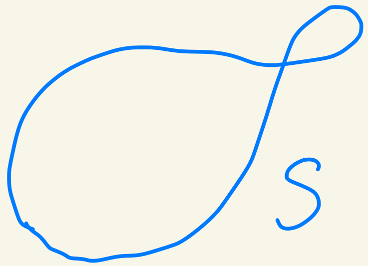
- For convenience let Σ connected, M, Σ oriented

Suppose $\exists$ immersed sphere $S \looparrowright M$

s.t. $F \cdot S \not\equiv S \cdot S \mod 2$



F



(i) $F \cdot S$ odd

$S \cdot S$ even

(ii) $F \cdot S$ even

$S \cdot S$ odd

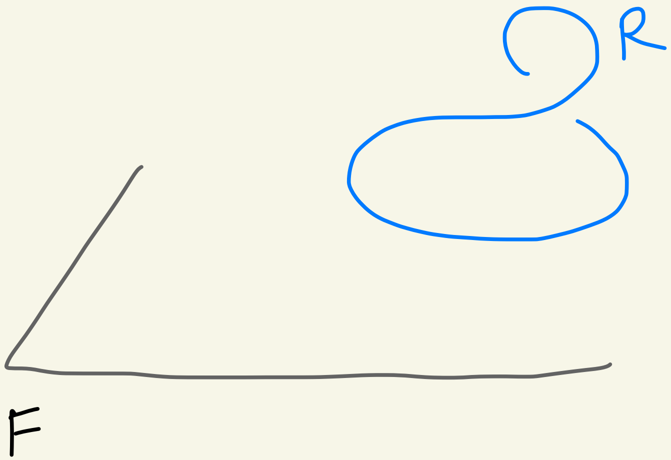
Otherwise, F is called s-characteristic.

When is $km(F; \{W_e\})$ independent of $\{W_e\}$?

- For convenience let Σ connected, M, Σ oriented

Suppose $\exists$ immersed $\mathbb{R}P^2$ $R \looparrowright M$

s.t. $F \cdot R \not\equiv R \cdot R \mod 2$



Otherwise, F is called r-characteristic.

When is $km(F;\{W_e\})$ independent of $\{W_e\}$?

- For convenience let Σ connected, M, Σ oriented

Let $\mathcal{B} \subseteq H_2(M, \Sigma^{cs}; \mathbb{Z}/2)$ the subset rep by maps of annuli or Möbius bands

Suppose the $\mathbb{Z}/2$ int form $\lambda_{\Sigma^{cs}}$ on $H_1(\Sigma^{cs}; \mathbb{Z}/2)$ is nontrivial on $\partial\mathcal{B}$

When is $\mathrm{km}(F; \{W_e\})$ independent of $\{W_e\}$?

- For convenience let Σ connected, M, Σ oriented

If $\lambda_{\Sigma^{cs}}|_{\partial B}$ trivial, define for a band B and A a collection of wh arcs for F^{cs}

$$\Theta_A(B) := \mu_{\Sigma^{cs}}(\partial B) + \partial B \pitchfork A + B \pitchfork F^{cs} + e(B) \mod 2$$

Suppose $\exists B$ s.t. $\Theta_A(B) \neq 0$

When is $km(F; \{W_e\})$ independent of $\{W_e\}$?

- For convenience let Σ connected, M, Σ oriented

Lemma : $\Theta_A(B)$ depends only on the homology class of B

If $\lambda_{\Sigma^{cs}}|_{\partial B} = 0$, Θ_A does not depend on A.

so there is a well defined map $\Theta : \mathcal{B} \longrightarrow \mathbb{Z}/2$

Definition : F is b-characteristic if $\lambda_{\Sigma^{cs}}|_{\partial B} = 0$ & $\Theta = 0$.