Slice knots which bound Klein bottles

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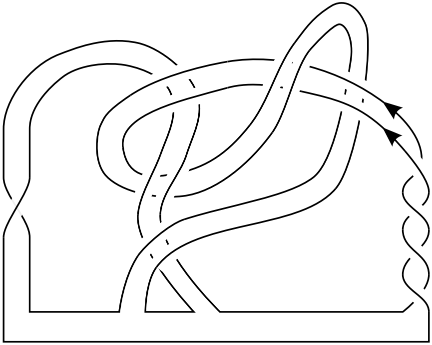
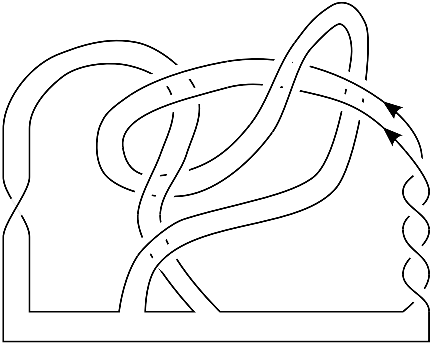
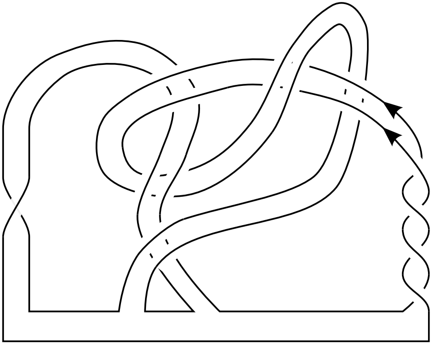




Theorem (R.)

If a slice knot K bounds a punctured Klein bottle F such that it has ‘zero framing’,

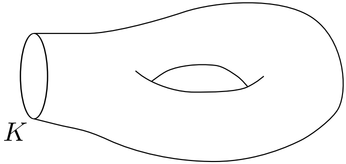


Theorem (R.)

If a slice knot K bounds a punctured Klein bottle F such that it has 'zero framing', we can find a 2-sided homologically essential simple closed curve J on F with self-linking zero which is slice in a $\mathbb{Z}\left[\frac{1}{2}\right]$-homology ball and hence, rationally slice (i.e. slice in a $\mathbb{Q}$-homology $\mathbb{B}^4$).

Introduction

Consider a knot K bounding a punctured torus F. Suppose we find a curve J which is homologically essential and has zero self-linking: we can surger the torus to get a slice disk for K. Such a curve on F is sometimes called a 'surgery curve' or 'derivative'.



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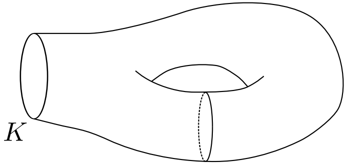

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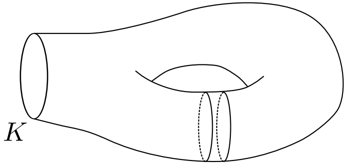

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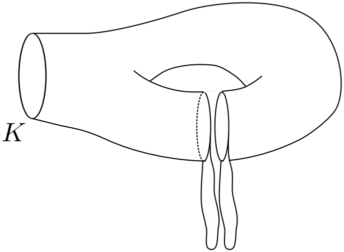

Kauffman's conjecture

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Conjecture (Kauffman, 1982)

If K is a slice knot and F is any genus one Seifert surface for K, there is a surgery curve J on F which is slice.

Slice knots of genus one

Theorem (Gilmer, 1983)

If K is algebraically slice and bounds a punctured torus F, then upto isotopy and orientation, there are exactly two homologically essential simple closed curves on F with zero self-linking.

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Evidence (Cooper, 1982)

If K is a genus one knot with $\Delta_K(t) \neq 1$, then at least one of the surgery curves (say J) satisfies

$$\sum_{i=0}^{r-1} \sigma_J(ca^i/p) = 0$$

where $m(m+1)$ is the leading term of $\Delta_K(t)$, $m \neq 0$, $c \in \mathbb{Z}_p^*$, $a = \frac{m+1}{m} \mod p$ and r is the order of a modulo p, for all p coprime to m and $m+1$.

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Evidence (Cochran-Davis, 2012)

There is a counterexample to Kauffman's conjecture, modulo the 4-dimensional Poincaré Conjecture.

Preliminaries

Suppose K bounds a punctured Klein bottle F. Let K^F be a pushoff of K into F.

Definition

We say that K *bounds* F *with zero framing* if $\text{lk}(K, K^F) = 0$.

Lemma (R.)

Given a knot K bounding a punctured Klein bottle F with zero framing, there exists a 2-sided homologically essential simple closed curve J on F such that

- *J has zero self-linking*
- *J is unique upto orientation and isotopy.*

J is the core of the 'orientation preserving band' if F is given in disk-band form.

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Suppose K bounds a punctured Klein bottle F with zero framing and has surgery curve J. If J is slice, so is K.

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Suppose K bounds a punctured Klein bottle F with zero framing and surgery curve J. Then $\sigma_K(\omega) = \sigma_J(\omega^2)$ for all $\omega \in \mathbb{S}^1$. In particular, if K is slice, $\sigma_J \equiv 0$

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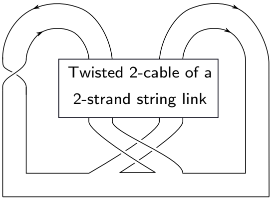
Proof: Such a K is concordant to $R(\eta, J)$, i.e. it is a satellite of J, where R is a ribbon knot.



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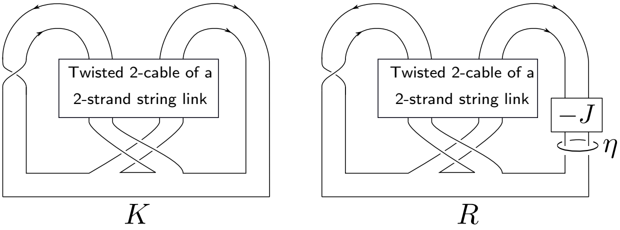

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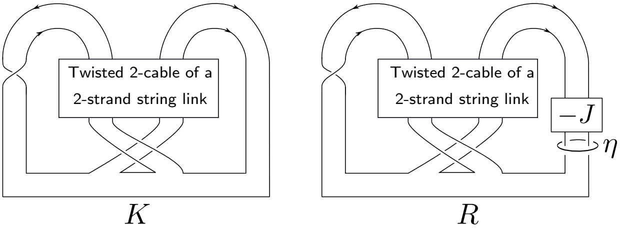

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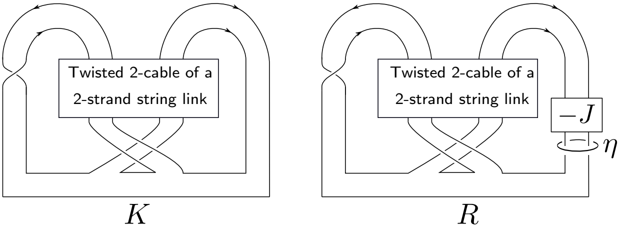

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Notice that if K is slice, $\sigma_J \equiv 0$. This is already more than the genus one case.

Main theorem

Theorem (R.)

Suppose a knot K bounds a punctured Klein bottle F with zero framing, and J is the surgery curve. K is $\mathbb{Z}\left[\frac{1}{2}\right]$-slice if and only if J is $\mathbb{Z}\left[\frac{1}{2}\right]$-slice.

(A knot is $\mathbb{Z}\left[\frac{1}{2}\right]$-slice if it bounds an embedded disk in a $\mathbb{Z}\left[\frac{1}{2}\right]$-homology $\mathbb{B}^4$.)
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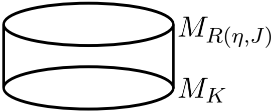
Note that in particular if K is slice, J is $\mathbb{Z}\left[\frac{1}{2}\right]$-slice.
Note also that the only known examples of $\mathbb{Z}\left[\frac{1}{2}\right]$-slice knots which are not also slice are satellites of strongly negatively amphichiral knots.

Proof:



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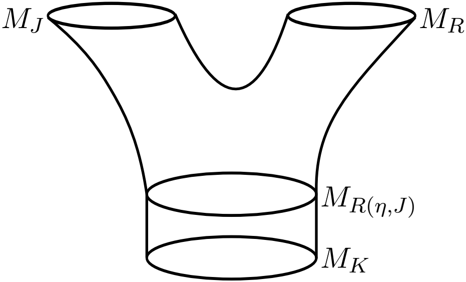

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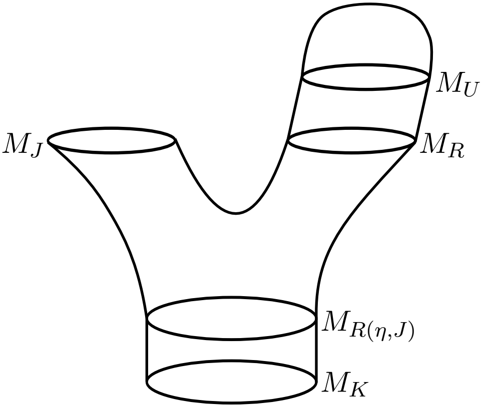


Here M_* denotes the zero-surgery manifold on the knot $*$

This gives a $\mathbb{Z}\left[\frac{1}{2}\right]$-homology cobordism between M_J and M_K.

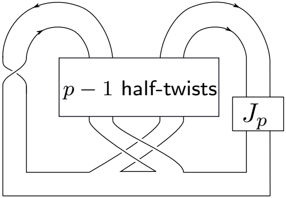
Theorem (Cochran-Franklin-Hedden-Horn, 2011)

M_K is smoothly $\mathbb{Z}\left[\frac{1}{2}\right]$-homology cobordant to M_U if and only if K is smoothly $\mathbb{Z}\left[\frac{1}{2}\right]$-slice.

$\square$

Interlude: an application

Hedden-Livingston-Ruberman (2011) used knots which bound Klein bottles as examples of topologically slice knots (not smoothly slice) which do not have Alexander polynomial one.



Here p is a prime number such that $p \equiv 3 \mod 4$ and J_p is the connected sum of $p-1$ copies of the untwisted double of the trefoil knot.

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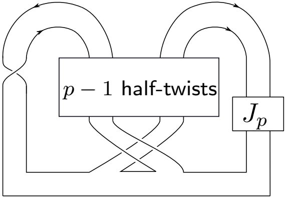

Here p is a prime number such that $p \equiv 3 \mod 4$ and J_p is the connected sum of $p-1$ copies of the untwisted double of the trefoil knot. Using our main theorem, we can quickly conclude that the above knots are not smoothly slice, since the knots J_p have non-zero τ-invariant.

Corollaries

Corollary (R.)

Given knots K and J, $K_{(2,p)}$ is $\mathbb{Z}\left[\frac{1}{2}\right]$-concordant to $J_{(2,p)}$ if and only if K is $\mathbb{Z}\left[\frac{1}{2}\right]$-concordant to J.

In particular, if $K_{(2,p)}$ is concordant to the $(2,p)$ torus knot, then K is $\mathbb{Z}\left[\frac{1}{2}\right]$-slice.

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Corollary (R.)

Given a knot K, if $K_{(2,1)}$ is $\mathbb{Z}\left[\frac{1}{2}\right]$-slice (or slice), then K is $\mathbb{Z}\left[\frac{1}{2}\right]$-slice.

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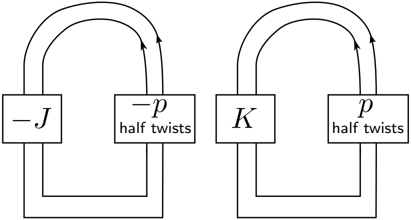

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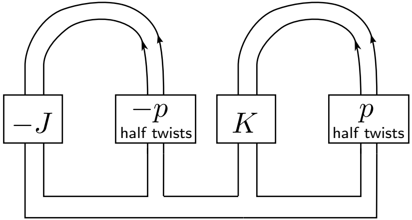

If As a result, $K_{(2,p)} \# (-J)_{(2,-p)}$ bounds a Klein bottle with 0 framing, with a disk band form where the orientation preserving band has knot type $K \# -J$. We can then apply our main theorem. □