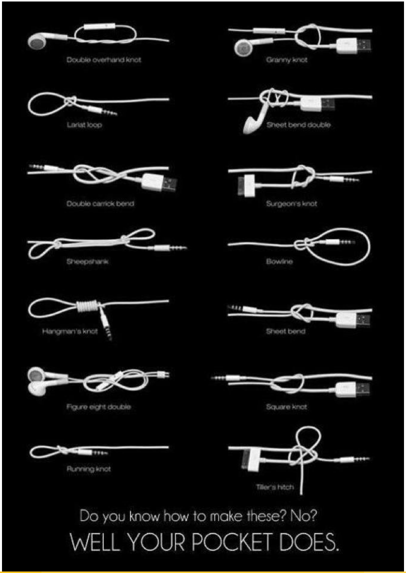
Knots, four dimensions, and fractals

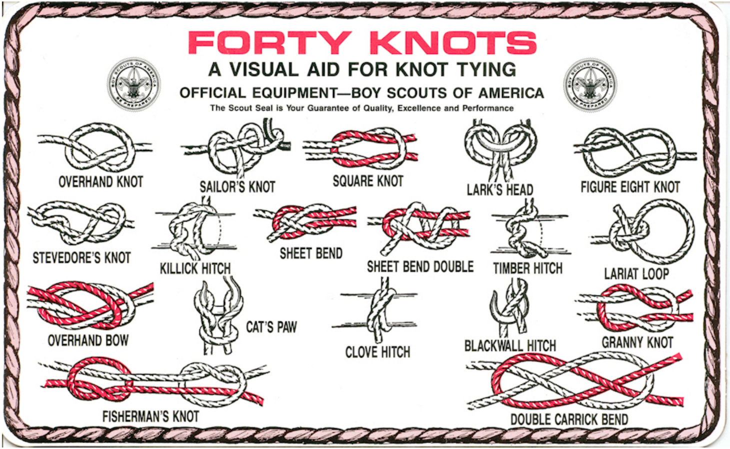
Arunima Ray
Brandeis University

February 6, 2017

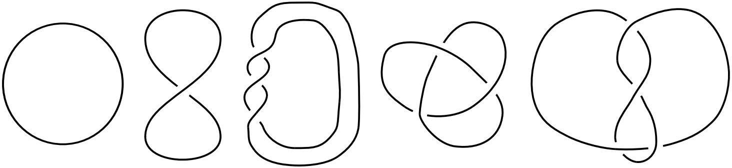
Examples of knots



Examples of knots



Mathematical knots



Take a piece of string, tie a knot in it, glue the two ends together.

Definition

A (mathematical) knot is a closed curve in space with no self-intersections.

Why knots?

Knot theory is a subset of the field of topology.

Theorem (Lickorish–Wallace, 1960s)

Any 3–dimensional 'manifold' can be obtained from $\mathbb{R}^3$ by performing an operation called 'surgery' on a collection of knots.

Why knots?

Knot theory is a subset of the field of topology.

Theorem (Lickorish–Wallace, 1960s)

Any 3–dimensional 'manifold' can be obtained from $\mathbb{R}^3$ by performing an operation called 'surgery' on a collection of knots.

Modern knot theory has applications to algebraic geometry, statistical mechanics, DNA topology, quantum computing, ...

Big questions in knot theory

1. How can we tell if two knots are equivalent?

Big questions in knot theory

1. How can we tell if two knots are equivalent?

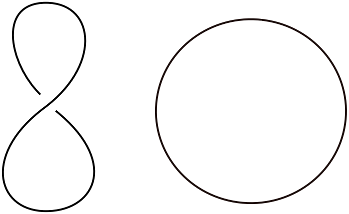


Figure: These are all pictures of the same knot!

Big questions in knot theory

1. How can we tell if two knots are equivalent?

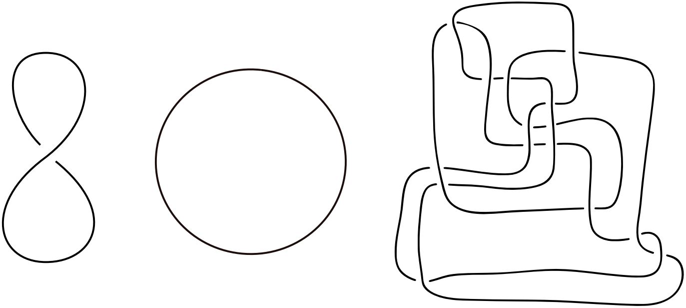


Figure: These are all pictures of the same knot!

Big questions in knot theory

1. How can we tell if two knots are equivalent?

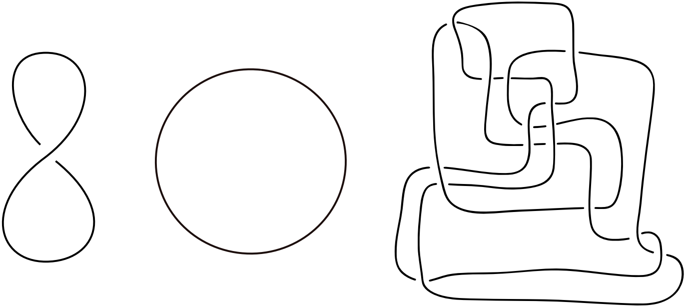


Figure: These are all pictures of the same knot!

2. How can we tell if two knots are distinct?

Big questions in knot theory

1. How can we tell if two knots are equivalent?

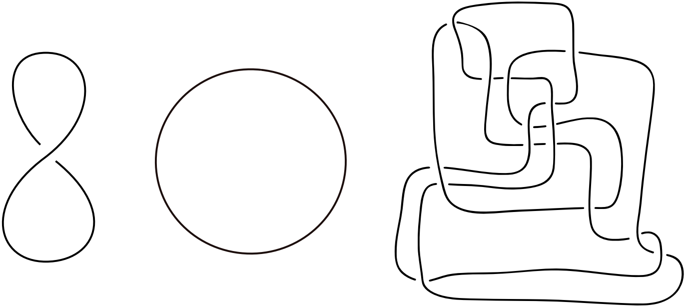


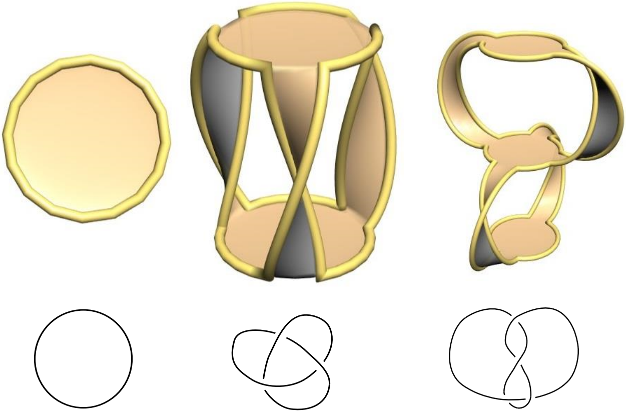
Figure: These are all pictures of the same knot!

2. How can we tell if two knots are distinct?
3. Can we quantify the 'knottedness' of a knot?

Genus of a knot

Proposition (Frankl–Pontrjagin, Seifert, 1930s)

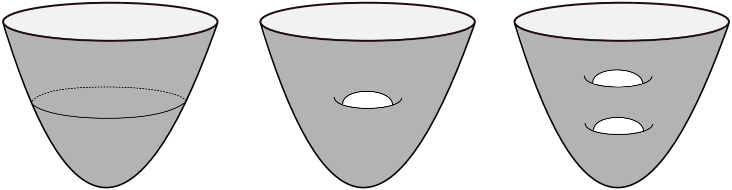
Any knot bounds a surface in $\mathbb{R}^3$.



Genus of a knot

Fundamental theorem in topology

Surfaces are classified by their genus.

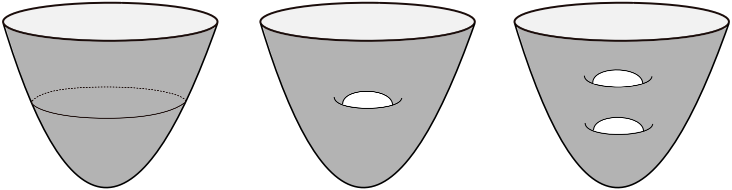


genus$= 0$ genus$= 1$ genus$= 2$

Genus of a knot

Fundamental theorem in topology

Surfaces are classified by their genus.



genus$=0$ genus$=1$ genus$=2$

Definition

The *genus* of a knot K, denoted $\mathrm{g}(K)$, is the least genus of surfaces bounded by K.

Genus of a knot

Proposition

If K and J are equivalent knots, then $g(K) = g(J)$.

Genus of a knot

Proposition

If K and J are equivalent knots, then $g(K) = g(J)$.

Proposition

A knot is the unknot if and only if it is the boundary of a disk.

That is, K is the unknot if and only if $g(K) = 0$.

Genus of a knot

Proposition

If K and J are equivalent knots, then $g(K) = g(J)$.

Proposition

A knot is the unknot if and only if it is the boundary of a disk.

That is, K is the unknot if and only if $g(K) = 0$.

If T is the trefoil knot, $g(T) = 1$. Therefore, the trefoil is not equivalent to the unknot.

Connected sum of knots

Figure: The connected sum of two trefoil knots, $T\#T$

Connected sum of knots

Figure: The connected sum of two trefoil knots, $T\#T$

Proposition

Given two knots K and J, $g(K\#J) = g(K) + g(J)$.

Connected sum of knots

Figure: The connected sum of two trefoil knots, $T \# T$

Proposition

Given two knots K and J, $g(K \# J) = g(K) + g(J)$.

Therefore, $g(\underbrace{T \# \cdots \# T}_{n \text{ copies}}) = n$

Connected sum of knots

Figure: The connected sum of two trefoil knots, $T\#T$

Proposition

Given two knots K and J, $\mathrm{g}(K\#J) = \mathrm{g}(K) + \mathrm{g}(J)$.

Therefore, $\mathrm{g}(\underbrace{T\#\cdots\#T}_{n \text{ copies}}) = n$

Corollary: There exist infinitely many distinct knots!

Connected sum of knots

Figure: The connected sum of two trefoil knots, $T\#T$

Proposition

Given two knots K and J, $\mathrm{g}(K\#J) = \mathrm{g}(K) + \mathrm{g}(J)$.

Therefore, $\mathrm{g}(\underbrace{T\#\cdots\#T}_{n \text{ copies}}) = n$

Corollary: There exist infinitely many distinct knots!

Corollary: We can never add together non-trivial knots to get a trivial knot.

Slice knots

Recall that a knot is equivalent to the unknot if and only if it is the boundary of a disk in $\mathbb{R}^3$.

Definition

A knot K is *slice* if it is the boundary of a disk in $\mathbb{R}^3 \times [0, \infty)$.

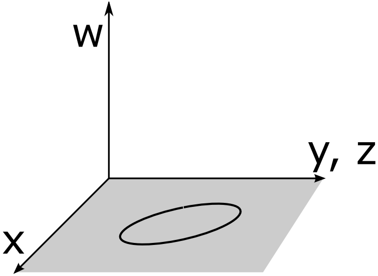


Figure: Schematic picture of the unknot

Slice knots

Recall that a knot is equivalent to the unknot if and only if it is the boundary of a disk in $\mathbb{R}^3$.

Definition

A knot K is *slice* if it is the boundary of a disk in $\mathbb{R}^3 \times [0, \infty)$.

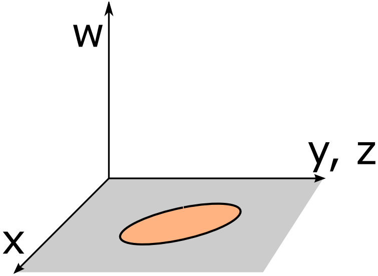


Figure: Schematic picture of the unknot

Slice knots

Recall that a knot is equivalent to the unknot if and only if it is the boundary of a disk in $\mathbb{R}^3$.

Definition

A knot K is *slice* if it is the boundary of a disk in $\mathbb{R}^3 \times [0, \infty)$.

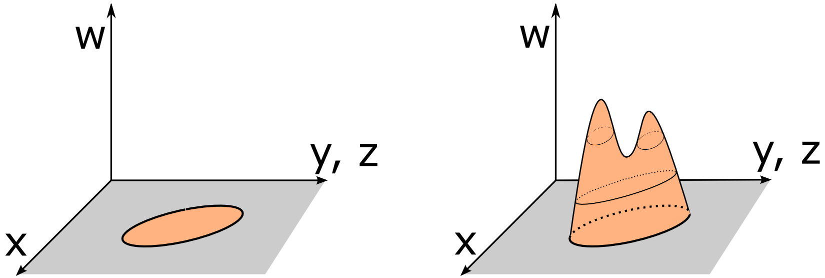
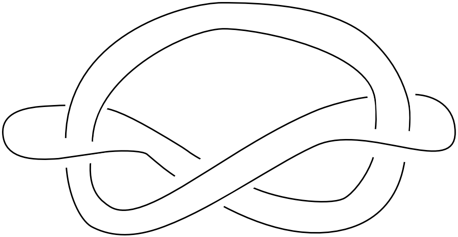
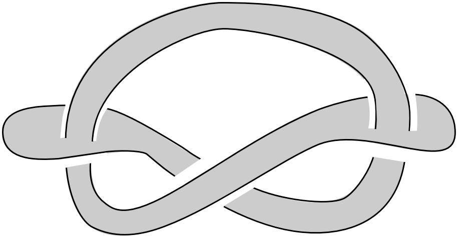


Figure: Schematic picture of the unknot and a slice knot

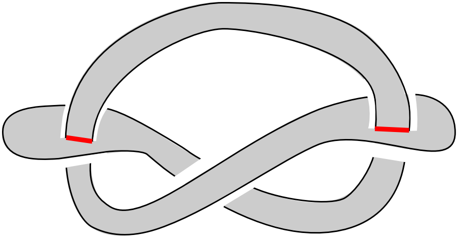
Examples of slice knots



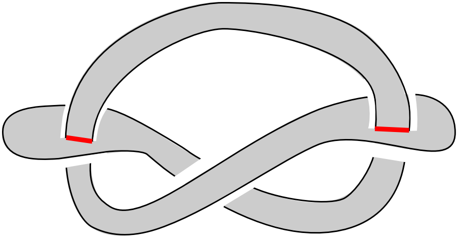
Examples of slice knots



Examples of slice knots

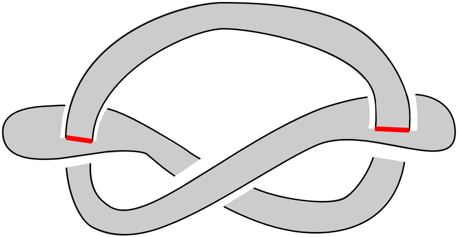


Examples of slice knots



Knots of this form are called *ribbon knots*.

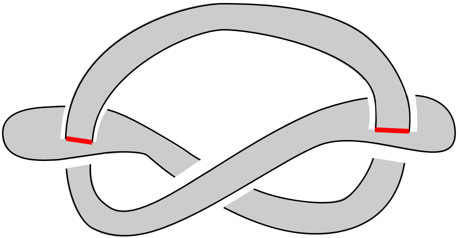
Examples of slice knots



Knots of this form are called *ribbon knots*.

Knots, modulo slice knots, form a group called the *knot concordance group*, denoted $\mathcal{C}$.

Examples of slice knots

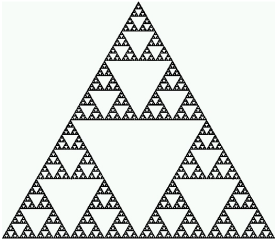


Knots of this form are called *ribbon knots*.

Knots, modulo slice knots, form a group called the *knot concordance group*, denoted $\mathcal{C}$. (The group operation is connected sum.)

Fractals

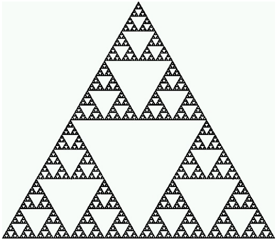
Fractals are objects that exhibit 'self-similarity' at arbitrarily small scales.



i.e. there exist families of injective functions from the set to smaller and smaller subsets (in particular, the functions are non-surjective).

Fractals

Fractals are objects that exhibit 'self-similarity' at arbitrarily small scales.



i.e. there exist families of injective functions from the set to smaller and smaller subsets (in particular, the functions are non-surjective).

Conjecture (Cochran–Harvey–Leidy, 2011)

The knot concordance group $\mathcal{C}$ is a fractal.

Satellite operations on knots

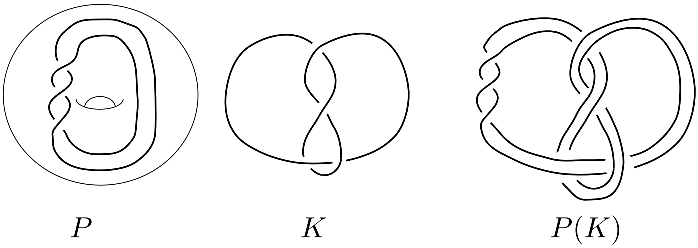


Figure: The satellite operation on knots

Satellite operations on knots

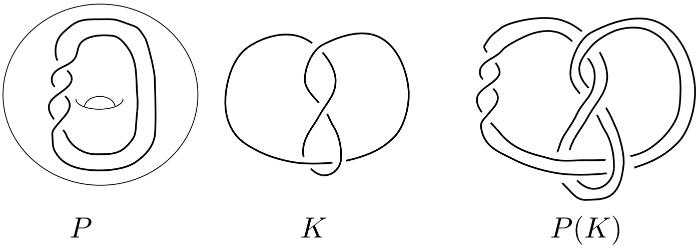


Figure: The satellite operation on knots

Any knot P in a solid torus gives a function on the knot concordance group,

$$\begin{aligned} P : \mathcal{C} &\to \mathcal{C} \\ K &\mapsto P(K) \end{aligned}$$

These functions are called *satellite operators*.

The knot concordance group has fractal properties

Theorem (Cochran–Davis–R., 2012)

Large (infinite) classes of satellite operators $P : \mathcal{C} \to \mathcal{C}$ *are injective.*

The knot concordance group has fractal properties

Theorem (Cochran–Davis–R., 2012)

Large (infinite) classes of satellite operators $P : \mathcal{C} \to \mathcal{C}$ are injective.

Theorem (R., 2013)

There are infinitely many satellite operators P and a large class of knots K such that $P^i(K) \neq P^j(K)$ for all $i \neq j$.

The knot concordance group has fractal properties

Theorem (Cochran–Davis–R., 2012)

Large (infinite) classes of satellite operators $P : \mathcal{C} \to \mathcal{C}$ *are injective.*

Theorem (R., 2013)

There are infinitely many satellite operators P *and a large class of knots* K *such that* $P^i(K) \neq P^j(K)$ *for all* $i \neq j$.

Theorem (Davis–R., 2013)

There exist satellite operators that are bijective on $\mathcal{C}$.

The knot concordance group has fractal properties

Theorem (Cochran–Davis–R., 2012)

Large (infinite) classes of satellite operators $P : \mathcal{C} \to \mathcal{C}$ *are injective.*

Theorem (R., 2013)

There are infinitely many satellite operators P *and a large class of knots* K *such that* $P^i(K) \neq P^j(K)$ *for all* $i \neq j$.

Theorem (Davis–R., 2013)

There exist satellite operators that are bijective on $\mathcal{C}$.

Theorem (A. Levine, 2014)

There exist satellite operators that are injective but not surjective.

Fractals

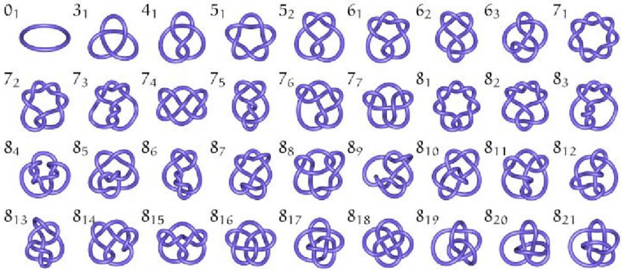
What is left to show?

In order for $\mathcal{C}$ to be a fractal, we need some notion of distance or size, to see that we have smaller and smaller embeddings of $\mathcal{C}$ within itself.

One way to do this is to exhibit a metric space structure on $\mathcal{C}$. There are several natural metrics on $\mathcal{C}$, but we have not yet found one that works well with the current results on satellite operators. The search is on!

The origins of mathematical knot theory

1880s: Kelvin (1824–1907) hypothesized that atoms were ‘knotted vortices’ in æther. This led Tait (1831–1901) to start tabulating knots.



Tait thought he was making a periodic table!

Examples of knots

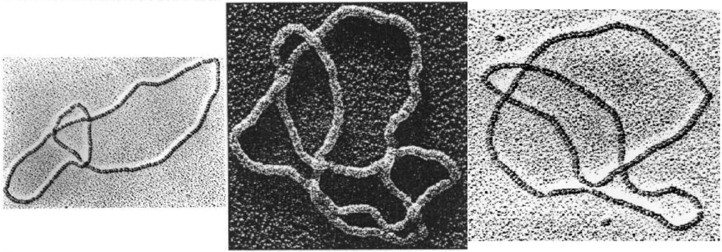


Figure: Knots in circular DNA.

(Images from Cozzarelli, Sumners, Cozzarelli, respectively.)