Casson towers and filtrations of the smooth knot concordance group

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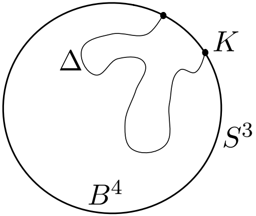
Definitions

Definition

A knot is *slice* if it bounds a smoothly embedded disk Δ in B^4.



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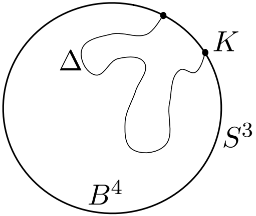
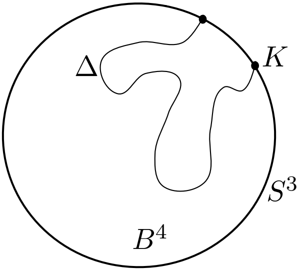


Knots, modulo slice knots, form the *smooth knot concordance group*, denoted $\mathcal{C}$.

There exist infinitely many smooth concordance classes of topologically slice knots (Endo, Gompf, Hedden–Kirk, Hedden–Livingston–Ruberman, Hom, etc.)

Approximating sliceness



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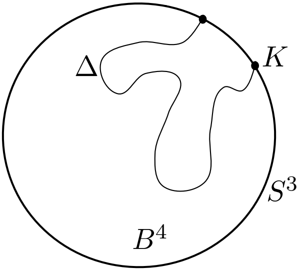

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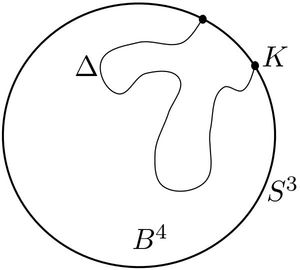

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The n–solvable filtration of $\mathcal{C}$

Definition (Cochran–Orr–Teichner, 2003)

For any $n \geq 0$, a knot K is in $\mathcal{F}_n$ (and is said to be n–*solvable*) if K bounds a smooth, embedded disk Δ in [[an approximation of B^4]]

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For any $n \geq 0$, a knot K is in $\mathcal{F}_n$ (and is said to be n*–solvable*) if K bounds a smooth, embedded disk Δ in a smooth, compact, oriented 4–manifold V with $\partial V = S^3$ such that

- $H_1(V) = 0$,
- there exist surfaces $\{L_1, D_1, L_2, D_2, \cdots, L_k, D_k\}$ embedded in $V - \Delta$ which generate $H_2(V)$ and with respect to which the intersection form is $\bigoplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,
- $\pi_1(L_i) \subseteq \pi_1(V - \Delta)^{(n)}$ for all i,
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Clearly,

$$\cdots \subseteq \mathcal{F}_n \subseteq \mathcal{F}_{n-1} \subseteq \cdots \subseteq \mathcal{F}_0 \subseteq \mathcal{C}$$

The n–solvable filtration of $\mathcal{C}$

- $\mathcal{F}_0 = \{K \mid \mathsf{Arf}(K) = 0\}$
- $\mathcal{F}_1 \subseteq \{K \mid K \text{ is algebraically slice}\}$
- $\mathcal{F}_2 \subseteq \{K \mid \text{various Casson–Gordon obstructions vanish}\}$

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$$\forall n,\ \mathbb{Z}^\infty \subseteq \mathcal{F}_n/\mathcal{F}_{n+1}$$

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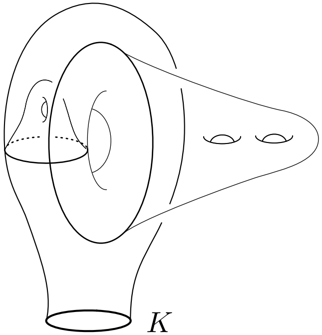

The grope filtration of $\mathcal{C}$

Model Theorem (Cochran–Orr–Teichner, 2003)

For all $n \geq 0$,

$$\mathcal{G}_{n+2} \subseteq \mathcal{F}_n$$

Topologically slice knots

Let $\mathcal{T}$ denote the set of all topologically slice knots.

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How can we use filtrations to study smooth concordance classes of topologically slice knots?

Positive and negative filtrations of $\mathcal{C}$

Definition (Cochran–Harvey–Horn, 2012)

For any $n \geq 0$, a knot K is in $\mathcal{P}_n$ (and is said to be n–*positive*) if K bounds a smooth, embedded disk Δ in [[an approximation of B^4]]

Positive and negative filtrations of $\mathcal{C}$

Definition (Cochran–Harvey–Horn, 2012)

For any $n \geq 0$, a knot K is in $\mathcal{P}_n$ (and is said to be n–*positive*) if K bounds a smooth, embedded disk Δ in a smooth, compact, oriented 4–manifold V with $\partial V = S^3$ such that

- $\pi_1(V) = 0$,
- there exist surfaces $\{S_i\}$ embedded in $V - \Delta$ which generate $H_2(V)$ and with respect to which the intersection form is $\bigoplus [1]$,
- $\pi_1(S_i) \subseteq \pi_1(V - \Delta)^{(n)}$ for all i,

Positive and negative filtrations of $\mathcal{C}$

Definition (Cochran–Harvey–Horn, 2012)

For any $n \geq 0$, a knot K is in $\mathcal{N}_n$ (and is said to be n–*negative*) if K bounds a smooth, embedded disk Δ in a smooth, compact, oriented 4–manifold V with $\partial V = S^3$ such that

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These filtrations can be used to distinguish smooth concordance classes of topologically slice knots

Goal

Prove a version of the result relating the grope filtration and n–solvable filtration, for the positive/negative filtrations

Casson towers

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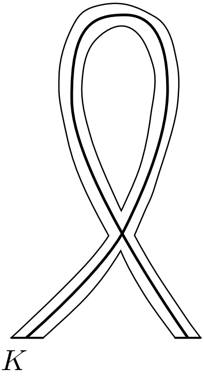
A Casson tower is built using layers of kinky disks, so they are natural objects to study in this context.

Casson towers



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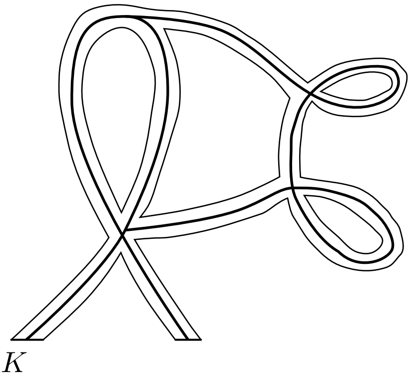

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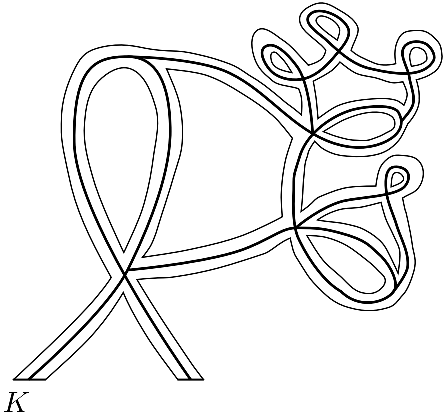

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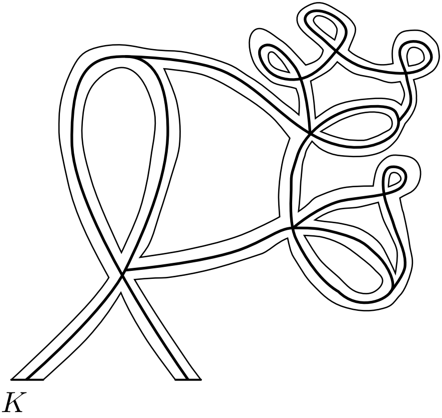

A Casson tower of height n consists of n layers of kinky disks.

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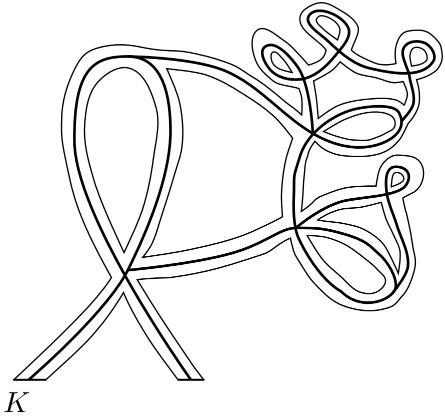

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A Casson tower T is of height $(2, n)$ if it has two layers of kinky disks, and each member of a standard set of generators of $\pi_1(T)$ is in $\pi_1(B^4 - T)^{(n)}$.

Casson towers

Definition (R.)

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- A knot is in $\mathfrak{C}_n$ if it bounds a Casson tower of height n in B^4
- A knot is in $\mathfrak{C}_n^+$ if it bounds a Casson tower of height n in B^4 such that all the kinks at the initial disk are positive
- A knot is in $\mathfrak{C}_n^-$ if it bounds a Casson tower of height n in B^4 such that all the kinks at the initial disk are negative

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- A knot is in $\mathfrak{C}_{2,n}$ if it bounds a Casson tower of height $(2, n)$ in B^4

Casson towers

Definition (R.)

- A knot is in $\mathfrak{C}_{2,\,n}$ if it bounds a Casson tower of height $(2,\,n)$ in B^4
- A knot is in $\mathfrak{C}^{+}_{2,\,n}$ if it bounds a Casson tower of height $(2,\,n)$ in B^4 such that all the kinks at the initial disk are positive
- A knot is in $\mathfrak{C}^{-}_{2,\,n}$ if it bounds a Casson tower of height $(2,\,n)$ in B^4 such that all the kinks at the initial disk are negative

Results

Model Theorem (Cochran–Orr–Teichner, 2003)

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- $\mathfrak{C}_{n+2} \subseteq \mathcal{G}_{n+2} \subseteq \mathcal{F}_n$
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Results

Proposition (R.)

For m–component links, let $\mathfrak{C}_n(m)$, $\mathfrak{C}_{2,n}(m)$, $\mathcal{F}_n(m)$, $\mathcal{P}_n(m)$, and $\mathcal{N}_n(m)$ denote the Casson tower, n–solvable, n–positive and n–negative filtrations respectively. For all n and $m \geq 2^{n+2}$,

$$\mathbb{Z} \subseteq \mathcal{F}_n(m)/\mathfrak{C}_{n+2}(m) \qquad \mathbb{Z} \subseteq \mathcal{F}_n(m)/\mathfrak{C}_{2,n}(m)$$

$$\mathbb{Z} \subseteq \mathcal{P}_n(m)/\mathfrak{C}^+_{n+2}(m) \qquad \mathbb{Z} \subseteq \mathcal{P}_n(m)/\mathfrak{C}^+_{2,n}(m)$$

$$\mathbb{Z} \subseteq \mathcal{N}_n(m)/\mathfrak{C}^-_{n+2}(m) \qquad \mathbb{Z} \subseteq \mathcal{N}_n(m)/\mathfrak{C}^-_{2,n}(m)$$

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