Satellite operations and fractal structures on knot concordance

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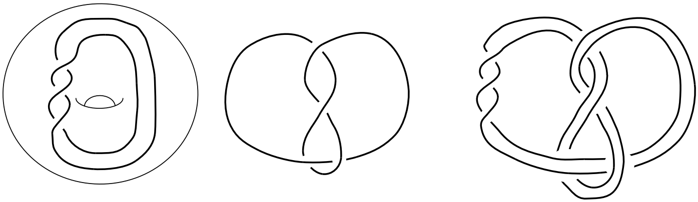
Satellite operations on knots



P K $P(K)$

Figure: The satellite operation on knots

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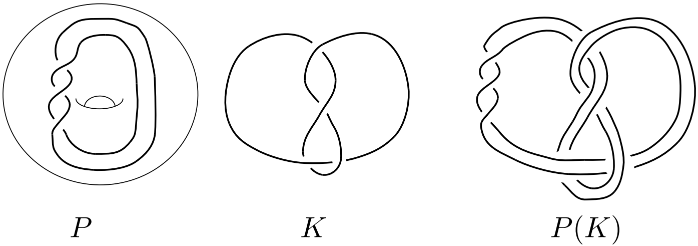


Figure: The satellite operation on knots

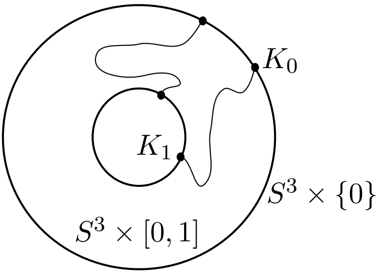
Any knot P in a solid torus gives a function on the set of knots.

$$\begin{aligned} P : \mathcal{K} &\to \mathcal{K} \\ K &\mapsto P(K) \end{aligned}$$

Knot concordance

Definition

Knots K_0, K_1 are *concordant* if they cobound a smoothly embedded annulus in $S^3 \times [0,1]$. Knots modulo concordance form the *knot concordance group* $\mathcal{C}$.

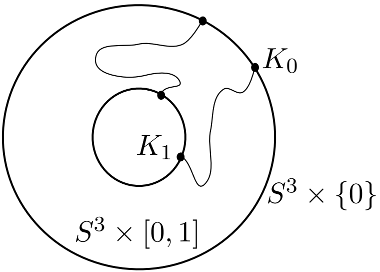


A knot is *slice* if it is concordant to the unknot.

Topological knot concordance

Definition

Knots K_0, K_1 are *topologically concordant* if they cobound a locally flat, topologically embedded annulus in $S^3 \times [0,1]$. Knots modulo topological concordance form the *topological knot concordance group* $\mathcal{C}_{\text{top}}$.

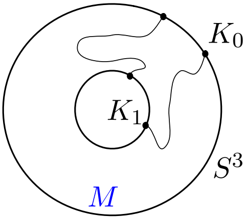


A knot is *topologically slice* if it is topologically concordant to the unknot.

Exotic knot concordance

Definition

Knots K_0, K_1 are *exotically concordant* if they cobound a smoothly embedded annulus in a smooth manifold M homeomorphic to $S^3 \times [0,1]$, i.e. a possibly exotic $S^3 \times [0,1]$. Knots modulo exotic concordance form the *exotic knot concordance group* $\mathcal{C}_{\text{ex}}$.



If the smooth 4–dimensional Poincaré Conjecture holds, then $\mathcal{C} = \mathcal{C}_{\text{ex}}$.

A knot is *exotically slice* if it is exotically concordant to the unknot.

Satellite operators on knot concordance

Any knot in a solid torus gives a well-defined map on knot concordance classes, called a *satellite operator*. That is, we have the following commutative diagram.

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{P} & \mathcal{K} \\ \downarrow & & \downarrow \\ \mathcal{C}_* & \xrightarrow{P} & \mathcal{C}_* \end{array}$$

for any $* \in \{\emptyset, \text{top}, \text{ex}\}$.

How do satellite operators act on knot concordance?

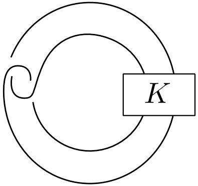


Figure: The untwisted Whitehead double of a knot K

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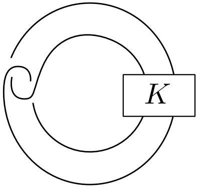

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Long-standing conjecture: $\mathrm{Wh}(K)$ slice $\Rightarrow K$ slice.

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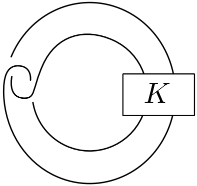

Figure: The untwisted Whitehead double of a knot K

Long-standing conjecture: $\mathrm{Wh}(K)$ slice $\Rightarrow K$ slice.
This can be restated as: what is the 'kernel' of $\mathrm{Wh} : \mathcal{C} \to \mathcal{C}$?

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1. is P 'weakly injective'? That is, if $P(K) = 0$, is $K = 0$?

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5. is P surjective?
6. what are the 'dynamics'?
7. any other question you might ask about functions.

Connected-sum

Connected-sum is a satellite operation.

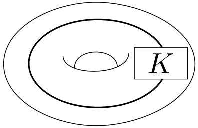

Figure: The pattern for connected-sum with the knot K

Connected-sum is both injective and surjective on any $\mathcal{C}_*$.

Previous results

Hedden (2007): if $\tau(K) > 0$, then $Wh^i(K)$ is not slice for any $i \geq 0$.

Cochran–Harvey–Leidy (2011): large classes of 'robust doubling operators' (winding number zero) injectively map large infinite subgroup of $\mathcal{C}$ to an independent set.

Hedden–Kirk (2012): the Whitehead doubling operator preserves the linear independence of an infinite independent set of torus knots.
(later generalized by Juanita Pinzón-Caicedo)

Injectivity of satellite operators

Theorem (Cochran–Davis–R.)

Any 'strong winding number ± 1' satellite operator is injective on $\mathcal{C}_{top}$ and $\mathcal{C}_{ex}$.

Thus, modulo smooth 4DPC, any strong winding number ± 1 satellite operator is injective on $\mathcal{C}$.

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Thus, modulo smooth 4DPC, any strong winding number ± 1 satellite operator is injective on $\mathcal{C}$.

Corollary: if $\tau(K) \neq 0$, then $P^i(K)$ is not slice for any winding number ± 1 satellite operator P with $P(U)$ slice, for any $i \geq 0$.

(There are analogous results for other non-zero winding numbers w, in terms of concordance in $\mathbb{Z}[\frac{1}{w}]$–homology $S^3 \times [0,1]$; in particular, any winding number ± 1 satellite operator is injective on concordance classes in integral homology $S^3 \times [0,1]$. For brevity, we will not discuss this much more.)

Strong winding number ± 1

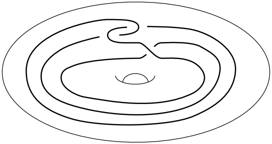

Figure: The Mazur pattern

Definition

A pattern P is 'strong winding number ± 1' if the meridian of the solid torus normally generates $\pi_1(S^3 - P(U))$.

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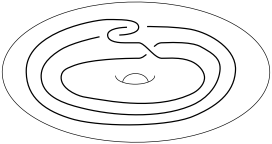
Strong winding number ± 1



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cf. P is winding number ± 1 if the meridian of the solid torus generates $H_1(S^3 - P(U))$.
If $P(U)$ is unknotted, strong winding number ± 1 is the same as winding number ± 1.

Proof of injectivity

First we prove weak injectivity for slice patterns.

Recall that a knot K is (topologically or exotically) slice if and only if the zero surgery M_K bounds a 4–manifold W where W is a homology circle and the meridian of K normally generates $\pi_1(W)$.

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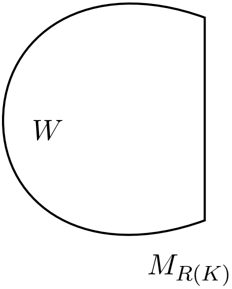
Lemma: If R is strong winding number ± 1 with $R(U)$ (topologically or exotically) slice then $M_{R(K)}$ is homology cobordant to M_K via a 4–manifold V where $\pi_1(V)$ is normally generated by the meridian of K.

Proof of injectivity

Now suppose that $R(K)$ is slice, $R(U)$ is slice, and R is strong winding number ± 1.



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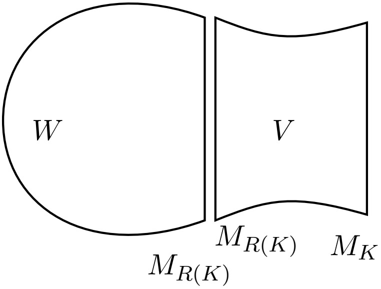

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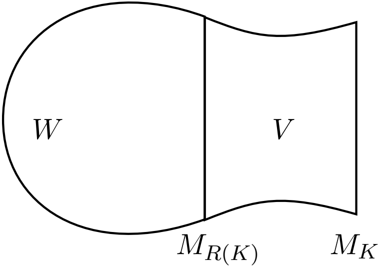

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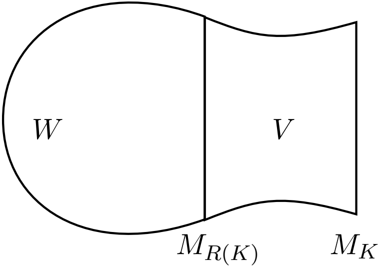

By the previous lemma, K is slice, and thus slice strong winding number ± 1 satellite operators are weakly injective.

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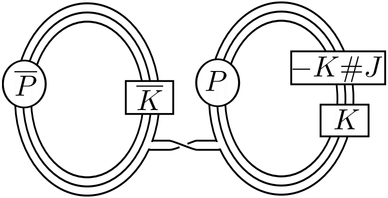
$$-P(K)\#[P(K\# - K\#J)] = 0$$

Proof of injectivity

We know that $-P(K)\#\left[P(K\#(-K\#J))\right]$ is slice. This knot is shown below.



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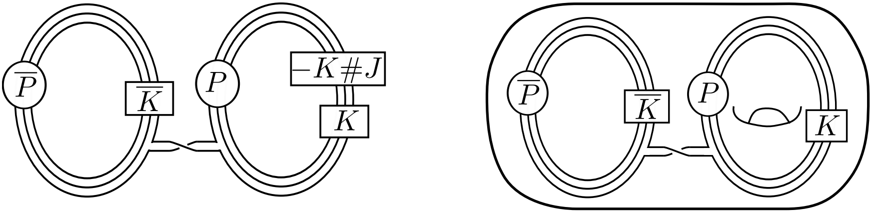

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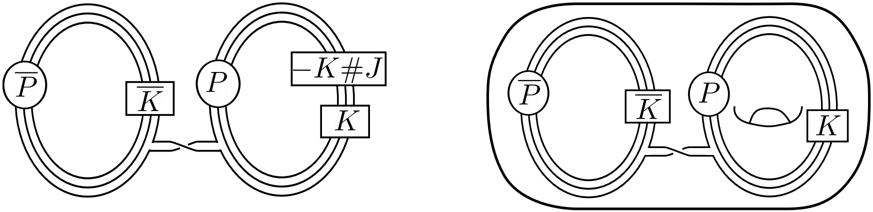

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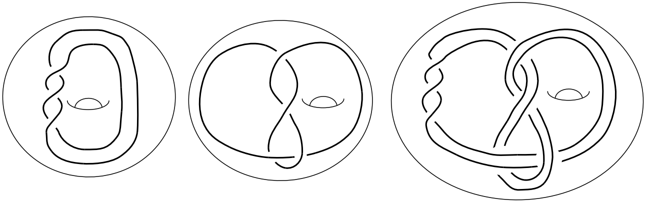
Thus, by weak injectivity for satellite operators with slice patterns, $-K\#J$ is slice, and thus $K = J$.

Satellite operators form a monoid



P Q $P \star Q$

Proposition

The satellite operation gives a monoid action on knots, i.e.

$$(P \star Q)(K) = P(Q(K))$$

Patterns and homology cylinders

Given a pattern P in a solid torus ST, let $E(P)$ denote the complement $ST - P$.

$E(P)$ is a 3–manifold with two toral boundary components, specifically a *homology cylinder*.

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Let $\widehat{\mathcal{S}}_*$ be the group of the 'strong' homology cylinders under 'strong' homology cobordism.

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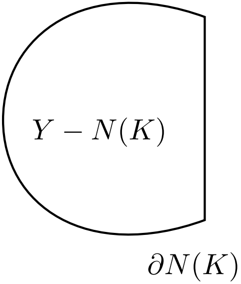
Let $\widehat{\mathcal{S}}_*$ be the group of the 'strong' homology cylinders under 'strong' homology cobordism.

There is a monoid homomorphism from the monoid of strong winding number ± 1 patterns to the group $\widehat{\mathcal{S}}_*$.

Homology cylinders act on knots in homology 3–spheres

Let V be a homology cylinder. Given a knot K in a homology 3–sphere Y, carve out $N(K)$, a solid torus neighborhood of K.



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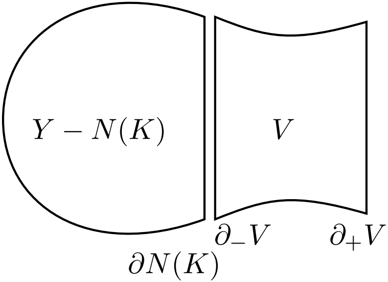

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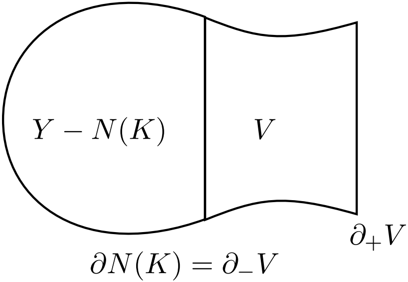

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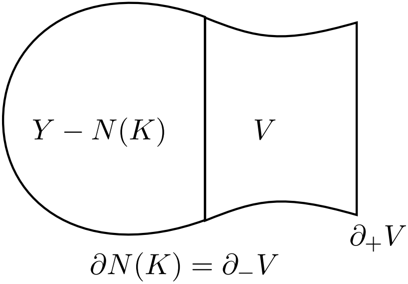

We obtain a 3–manifold with a single torus boundary component. We can canonically glue in a solid torus to get a homology 3–sphere. The core of this solid torus is the new knot.

Generalizations of knot concordance

Let $\widehat{\mathcal{C}}_*$ be the group of knots in homology spheres modulo concordance in 'strong' homology cobordisms.

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There are injective homomorphisms $\mathcal{C}_* \hookrightarrow \widehat{\mathcal{C}}_*$.

(Davis–R.): $\widehat{\mathcal{S}}_*$ acts on $\widehat{\mathcal{C}}_*$ by a group action.

Satellite operators as group actions

Theorem (Davis–R.)

For $* =$ ex or top, and any strong winding number one satellite operator P, the following diagram commutes.

$$\begin{array}{ccc} \mathcal{C}_* & \xrightarrow{P} & \mathcal{C}_* \\ \big\downarrow & & \big\downarrow \\ \widehat{\mathcal{C}}_* & \xrightarrow{E(P)} & \widehat{\mathcal{C}}_* \end{array}$$

Since $\widehat{\mathcal{S}}_*$ gives a group action on $\widehat{\mathcal{C}}_*$, each $E(P) \in \widehat{\mathcal{S}}_*$ acts via a bijection. The Cochran–Davis–R. injectivity result for strong winding number ± 1 satellite operators follows.

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Theorem (Davis–R.)

Let $P \subseteq ST = S^1 \times D^2$ be winding number one. If the meridian of P is in the normal subgroup of $\pi_1(E(P))$ generated by the meridian of ST, then P is strong winding number one and there exists a strong winding number one pattern $\overline{P}$ such that $E(\overline{P}) = E(P)^{-1}$ as homology cylinders.

In particular, $\overline{P}(P(K))$ is (exotically or topologically) concordant to K for any knot K.

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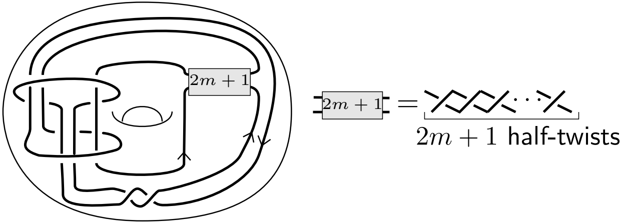
In particular, $\overline{P}(P(K))$ is (exotically or topologically) concordant to K for any knot K.

Consequently, $P : \mathcal{C}_* \to \mathcal{C}_*$ is a bijection.

Bijective satellite operators

For each $m \geq 0$, the satellite operator P_m shown below has an inverse satellite operator $\overline{P_m}$ which can be explicitly drawn, i.e. $\overline{P_m}(P_m(K))$ is concordant to K for any knot K. Moreover, each $P_m : \mathcal{C}_* \to \mathcal{C}_*$ is bijective and P_m is distinct from all connected-sum operators in $\widehat{\mathcal{S}}_*$.



Note that it is still possible that, for some fixed knot J, $P_m(K) = J\#K$ for all K, i.e. it is not known whether patterns act faithfully.

Non-surjectivity of satellite operators

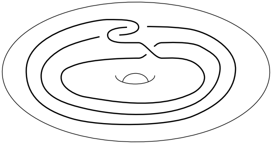


Figure: The Mazur pattern

In contrast, recall from yesterday that the Mazur satellite operator is non-surjective on $\mathcal{C}$ (A. Levine).
In particular, Levine showed that no knot J with $\varepsilon(J) = -1$ is in the image of the Mazur satellite operator.

Note that it is not known whether the Mazur satellite operator is the identity function on $\mathcal{C}_{\text{top}}$.

Other results

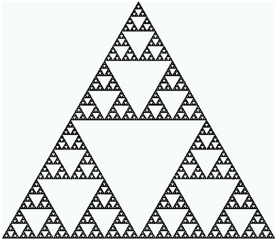
K. Park: $Wh(T_{2,2m+1})$ and $Wh^2(T_{2,2m+1})$ generate a $\mathbb{Z} \oplus \mathbb{Z}$ summand of the subgroup of topologically slice knots in $\mathcal{C}$.

R. : For several classes of strong winding number ± 1 patterns P (including the Mazur pattern) and infinitely many knots K, $P^i(K) \neq P^j(K)$ in $\mathcal{C}_{\text{ex}}$ for any $i \neq j \geq 0$.
(For the Mazur pattern, this can be improved by A. Levine's computation of τ–invariants.)

Feller–J. Park–R. : Let M be the Mazur satellite operator. There exists an infinite family of topologically slice knots $\{K_i\}$ such that for all $r \geq 0$, $\{M^r(K_i)\}$ generates a subgroup of $\mathcal{C}$ of infinite rank.

Fractals

Fractals are objects that exhibit 'self-similarity' at arbitrarily small scales.



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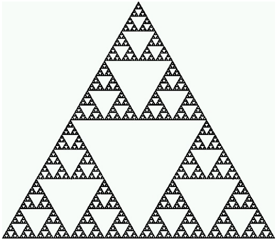
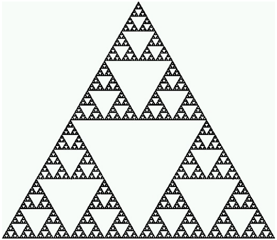


i.e. there exist families of injective functions from the set to smaller and smaller subsets (in particular, the functions are non-surjective).

Fractals

Fractals are objects that exhibit 'self-similarity' at arbitrarily small scales.



i.e. there exist families of injective functions from the set to smaller and smaller subsets (in particular, the functions are non-surjective).

Conjecture (Cochran–Harvey–Leidy, 2011)

The knot concordance group $\mathcal{C}$ is a fractal.

The knot concordance group has fractal properties

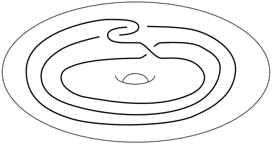


Figure: The Mazur pattern M

Cochran–Davis–R. : M is injective on $\mathcal{C}_{\text{ex}}$ and $\mathcal{C}_{\text{top}}$.

A. Levine: M is not surjective on $\mathcal{C}$. Moreover,

$$Im(M) \supsetneq Im(M^2) \supsetneq Im(M^3) \supsetneq \cdots$$

What about scale?

The knot concordance group has fractal properties

To properly address the question of scale we need some notion of distance on $\mathcal{C}_*$. This was started by Cochran–Harvey, with further work by Cochran–Harvey–Powell (see talk on Saturday).