Columbia geometry and topology seminar
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Counterexamples in 4-manifold topology

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Counterexamples in 4-manifold topology

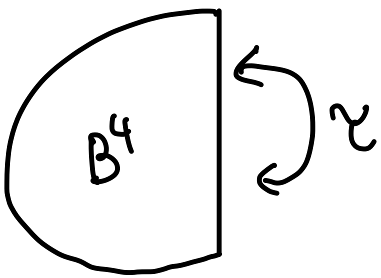
$$\boxed{\text{Diffeomorphic}} \longrightarrow \boxed{\text{Homeomorphic}} \longrightarrow \boxed{\text{htpy equiv}}$$

Manifolds will be closed, connected.

Counterexamples in 4-manifold topology

$$\boxed{\text{Diffeomorphic}} \;\underset{\text{exotic}}{\overset{\longrightarrow}{\not\longleftarrow}}\; \boxed{\text{Homeomorphic}} \;\underset{\text{fake}}{\overset{\longrightarrow}{\not\longleftarrow}}\; \boxed{\text{htpy equiv}}$$

$\mathbb{RP}^4 := B^4 \Big/ {\scriptstyle x \sim \tau(x) \atop \scriptstyle x \in \partial B^4}$

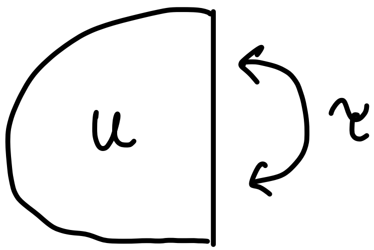


$\exists\, \tau : \partial B^4 \circlearrowleft$ sm, free involution

(antipodal map)

Cappell-Shaneson
Fintushel-Stern

$R := \mathcal{U} \Big/ {\scriptstyle x \sim \tau(x) \atop \scriptstyle x \in \partial \mathcal{U}}$



$R \not\cong \mathbb{RP}^4$

$R \approx \mathbb{RP}^4$

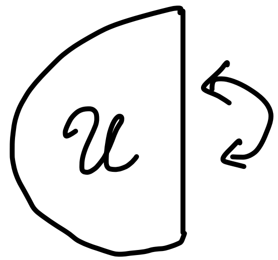
$\mathcal{U}$ sm, compact, contractible,

$\partial \mathcal{U} \cong \Sigma(3,5,19)$

$\exists\, \tau : \partial \mathcal{U} \circlearrowleft$ sm, free inv.

Ruberman

$R := \mathcal{U} \Big/ {\scriptstyle x \sim \tau(x) \atop \scriptstyle x \in \partial \mathcal{U}}$



$R \simeq \mathbb{RP}^4$

$R \not\approx \mathbb{RP}^4$

$\mathcal{U}$ top, compact, contr

$\partial \mathcal{U} \cong \Sigma(5,9,13)$

$\exists\, \tau : \partial \mathcal{U} \circlearrowleft$ sm, free involution

$KS(R) \neq 0 \Rightarrow R$ not smoothable

Counterexamples in 4-manifold topology

$$\begin{tikzcd}[column sep=huge] \boxed{\text{Diffeomorphic}} \arrow[r, shift left] & \boxed{\text{Homeomorphic}} \arrow[l, shift left, "\times" description, "{\mathbb{RP}^4,\ \mathbb{R}}"] \arrow[r, shift left] & \boxed{\text{htpy equiv}} \arrow[l, shift left, "\times" description, "{\mathbb{RP}^4,\ \mathbb{R}}"] \end{tikzcd}$$

Other examples of exotic mflds.

$E(1)$, $E(1)_{2,3}$ [Donaldson]

$\cong$

$\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$

Simply connected $\Rightarrow$ orientable

$E(2)_{2,q}$, q odd

K3 surface admits infinitely many sm str. [Fintushel Stern]

Problem: Find exotic mflds with low euler characteristic

e.g. $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2$ has a sm. str. [Akhmedov Park]

Counterexamples in 4-manifold topology

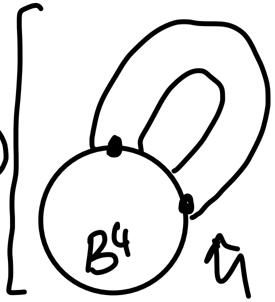
$$\begin{tikzcd}
\boxed{\text{Diffeomorphic}} \arrow[r, shift left] & \boxed{\text{Homeomorphic}} \arrow[l, shift left, "\times"] \arrow[r, shift left] & \boxed{\text{htpy equiv}} \arrow[l, shift left, "\times"]
\end{tikzcd}$$

RP^4, R
$E(1)$, $E(1)_{2,3}$
K3 surface, ∞

RP^4, R

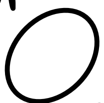
$\mathbb{C}P^2 := X_1(0) \cup B^4$

$X_1(0)$



attach a 2-handle $D^2 \times D^2$ w. framing $+1$

O unknot



$\subseteq S^3$

$\partial X_1(0) = S^3_1(0) = S^3$

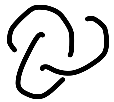
$*\mathbb{C}P^2 := X_1(RHT) \cup C^4$

Chern manifold [Freedman]

$X_1(RHT)$

$D^2 \times D^2$ with framing $+1$

RHT



$*\mathbb{C}P^2 \simeq \mathbb{C}P^2$ [Wall]

$KS(*\mathbb{C}P^2) \neq 0$

$\Rightarrow$ not smoothable

$\partial X_1(RHT) = S^3_1(RHT) = \overset{P}{\text{Poincare hom. sphere}}$

[Freedman] $P = \partial C^4$, C compact contractible

Counterexamples in 4-manifold topology

$$\begin{tikzcd} \boxed{\text{Diffeomorphic}} \arrow[r, shift left] & \boxed{\text{Homeomorphic}} \arrow[l, shift left, "\times"] \arrow[r, shift left] & \boxed{\text{htpy equiv}} \arrow[l, shift left, "\times"] \end{tikzcd}$$

Homeomorphic ⇏ Diffeomorphic:
RP^4, R
$E(1),\ E(1)_{2,3}$
K3 surface, ∞

htpy equiv ⇏ Homeomorphic:
RP^4, R
$CP^2, *CP^2$
smoothable?

$\mathbb{R}P^4 \# \mathbb{C}P^2$ and $R \# *\mathbb{C}P^2$

↖ smoothable [Ruberman-Stern]

Fact: $\exists$ knot K s.t. $S^3_1(K) \cong \Sigma(5,9,13)$

$X_1(K)$

$D^2 \times D^2$ to K
with framing $+1$

$\partial X_1(K) = S^3_1(K) = \Sigma(5,9,13)$ ↺ τ

$$R \# *\mathbb{C}P^2 \approx X_1(K) \Big/ \begin{array}{l} x \sim \tau(x) \\ x \in \partial X_1(K) \end{array}$$

$R \# *\mathbb{C}P^2 \not\approx \mathbb{R}P^4 \# \mathbb{C}P^2$ [Hambleton-Kreck-Teichner]

Counterexamples in 4-manifold topology

$$\begin{tikzcd}
\boxed{\text{Diffeomorphic}} \arrow[r, shift left] & \boxed{\text{Homeomorphic}} \arrow[l, shift left, "\times"] \arrow[r, shift left] & \boxed{\text{htpy equiv}} \arrow[l, shift left, "\times"]
\end{tikzcd}$$

$\mathbb{R}P^4$, R	$\mathbb{R}P^4$, R
$E(1)$, $E(1)_{2,3}$	$\mathbb{C}P^2$, $*\mathbb{C}P^2$
K3 surface, ∞	$\mathbb{R}P^4 \# \mathbb{C}P^2$, $R \# *\mathbb{C}P^2$

Do $\exists$ sm, orientable, simply connected, fake 4-mflds? No.
[Wall + Freedman]

Counterexamples in 4-manifold topology

$$\begin{tikzcd}[column sep=huge]
\boxed{\text{Diffeomorphic}} \arrow[r, shift left] & \boxed{\text{Homeomorphic}} \arrow[l, shift left, "\times"', "\begin{array}{c} \mathbb{RP}^4,\ R \\ E(1),\ E(1)_{2,3} \\ K3 \text{ surface}, \infty \end{array}"] \arrow[r, shift left] & \boxed{\text{htpy equiv}} \arrow[l, shift left, "\times"', "\begin{array}{c} \mathbb{RP}^4,\ R \\ \mathbb{CP}^2,\ *\mathbb{CP}^2 \\ \mathbb{RP}^4 \# \mathbb{CP}^2,\ R \# *\mathbb{CP}^2 \end{array}"]
\end{tikzcd}$$

Do $\exists$ sm, orientable, ~~simply connected~~, fake 4-mflds?

Let lens spaces L, L' s.t. $L \simeq L'$ but $L \not\cong L'$.

[Turaev] $L \times S^1 \simeq L' \times S^1$ but $L \times S^1 \not\cong L' \times S^1$.

Counterexamples in 4-manifold topology

$$\begin{tikzcd} \boxed{\text{Diffeomorphic}} \arrow[r, shift left] & \boxed{\text{Homeomorphic}} \arrow[l, shift left, "\times"] \arrow[r, shift left] & \boxed{\text{htpy equiv}} \arrow[l, shift left, "\times"] \end{tikzcd}$$

$\mathbb{R}P^4, R$

$E(1),\ E(1)_{2,3}$

K3 surface, ∞

$\mathbb{R}P^4, R$

$\mathbb{C}P^2,\ *\mathbb{C}P^2$

$\mathbb{R}P^4 \# \mathbb{C}P^2,\ R \# *\mathbb{C}P^2$

$L \times S^1,\ L' \times S^1;\ L \simeq L',\ L \not\cong L'$

Do $\exists$ infinitely many sm, orientable, fake 4-mflds? Open.

$\exists$ ∞ many TOP orientable, fake 4-mflds.

$p \geq 3$ odd $\quad L_{p,q} \times S^1$

$\exists$ ∞ly many $M \simeq L_{p,q} \times S^1$, $M \not\cong L_{p,q} \times S^1$.

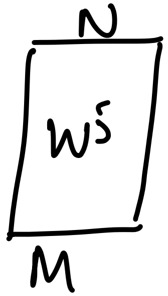
[Kwasik-Schultz]

Counterexamples in 4-manifold topology

$$
\begin{tikzcd}
\boxed{\text{Diffeomorphic}} \arrow[r, shift left] \arrow[d] & \boxed{\text{Homeomorphic}} \arrow[l, shift left, "\times" description, "\text{exotic}"] \arrow[d] & \\
\boxed{\text{Sm. s-cobordant}} \arrow[r] \arrow[d] & \boxed{\text{TOP s-cobordant}} \arrow[r] \arrow[d] & \boxed{\text{simple htpy equiv}} \arrow[d] \\
\boxed{\text{Sm. h-cobordant}} \arrow[r] & \boxed{\text{TOP h-cobordant}} \arrow[r] & \boxed{\text{htpy equiv}}
\end{tikzcd}
$$

A htpy equiv $f: X \to Y$ is simple if its whitehead torsion $\tau(f) \in Wh(\pi_1 X)$ is trivial.

M^4, N^4 are h-cobordant if $\exists$



s.t. $\underbrace{\begin{matrix} M \hookrightarrow W \\ N \hookrightarrow W \end{matrix} \text{ htpy equiv.}}_{\text{if simple htpy equiv, then } W \text{ is an s-cob.}}$

Counterexamples in 4-manifold topology

$$\begin{tikzcd}
\text{Diffeomorphic} \arrow[r, shift left] \arrow[d] & \text{Homeomorphic} \arrow[l, shift left, "\times"] \arrow[d] & \\
\text{Sm. s-cobordant} \arrow[r] \arrow[d] & \text{TOP s-cobordant} \arrow[r] \arrow[d] & \text{simple htpy equiv} \arrow[d] \\
\text{Sm. h-cobordant} \arrow[r] & \text{TOP h-cobordant} \arrow[r, shift left] & \text{htpy equiv} \arrow[l, shift left, "\times"]
\end{tikzcd}$$

$RP^4, \mathcal{R}$

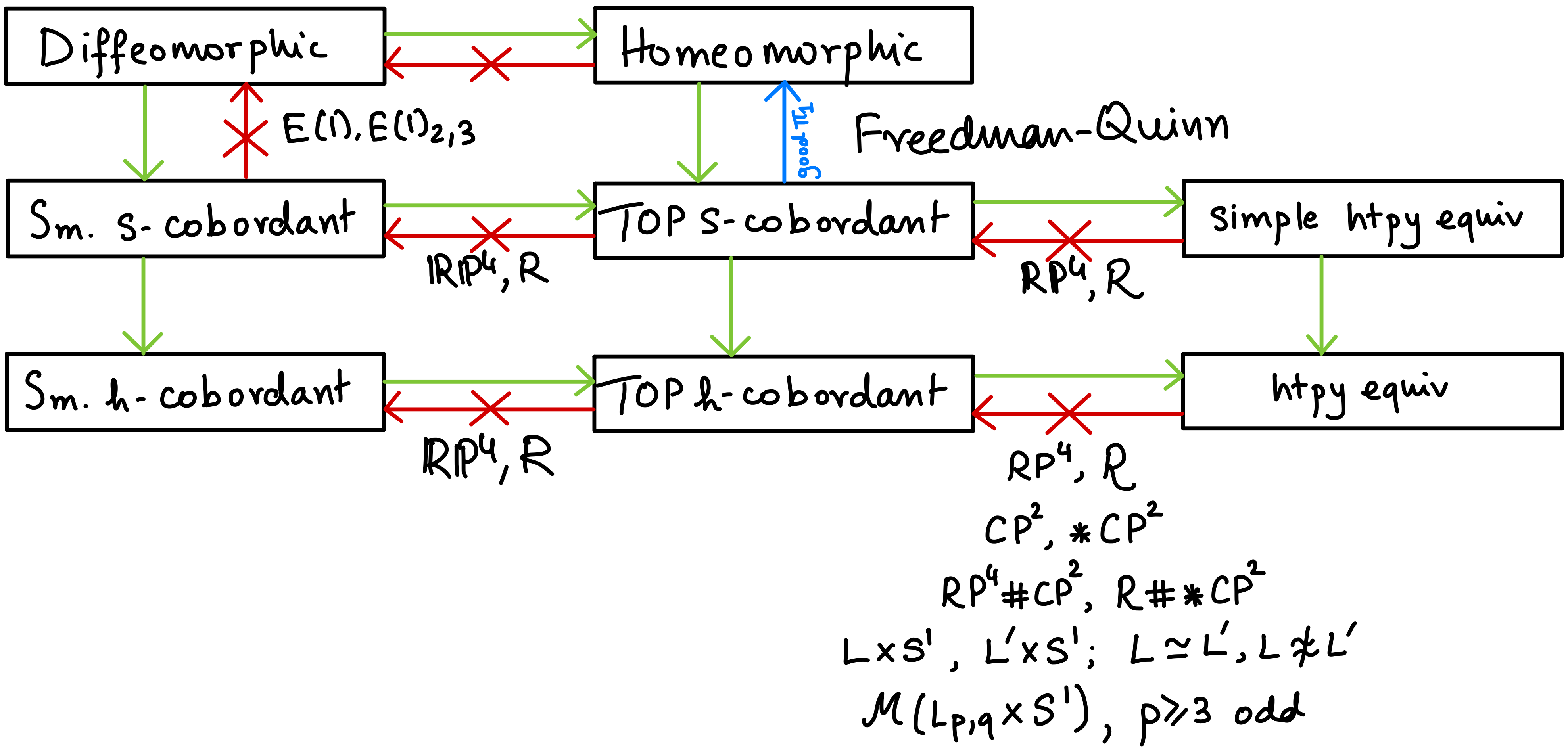
$CP^2, *CP^2$

$RP^4 \# CP^2, \; \mathcal{R} \# *CP^2$

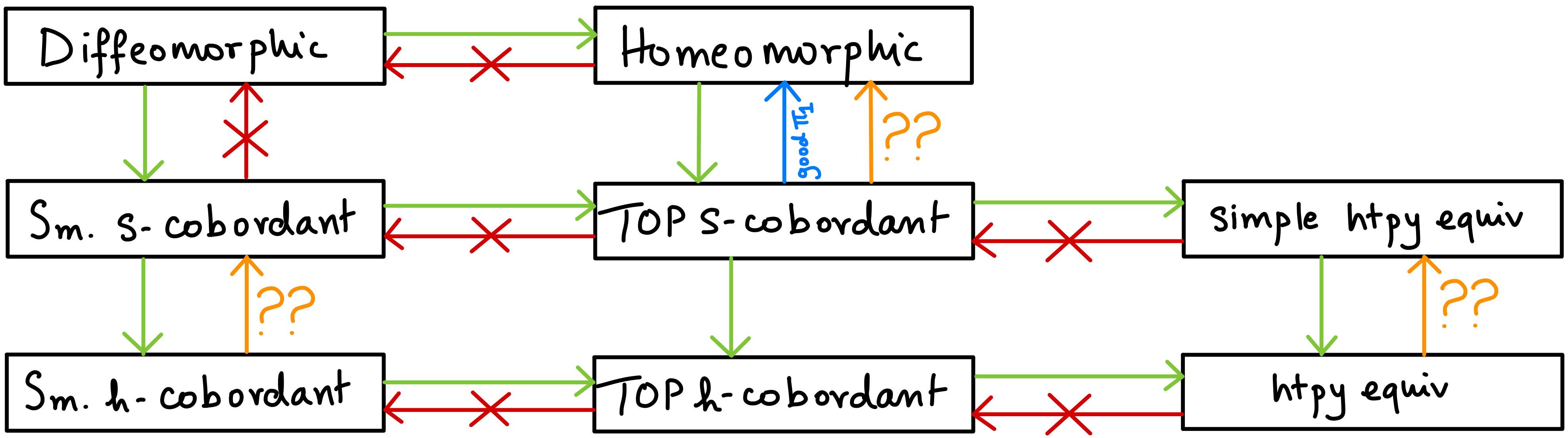
$L \times S^1, \; L' \times S^1; \quad L \simeq L', \; L \not\cong L'$

$\mathcal{M}(L_{p,q} \times S^1), \; p \geqslant 3$ odd

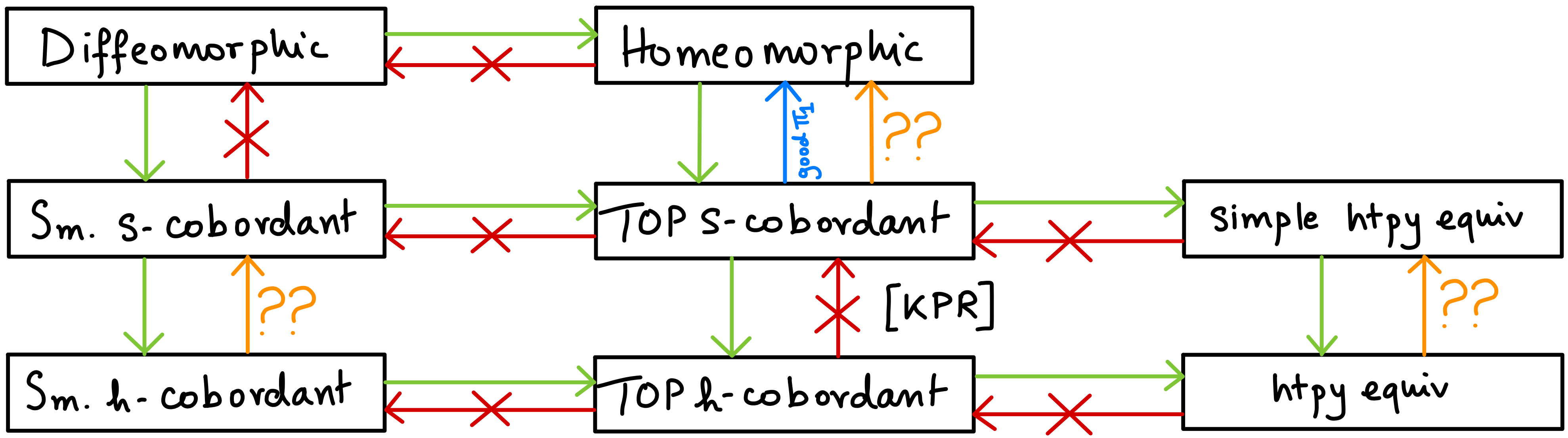
Counterexamples in 4-manifold topology



Counterexamples in 4-manifold topology



Counterexamples in 4-manifold topology



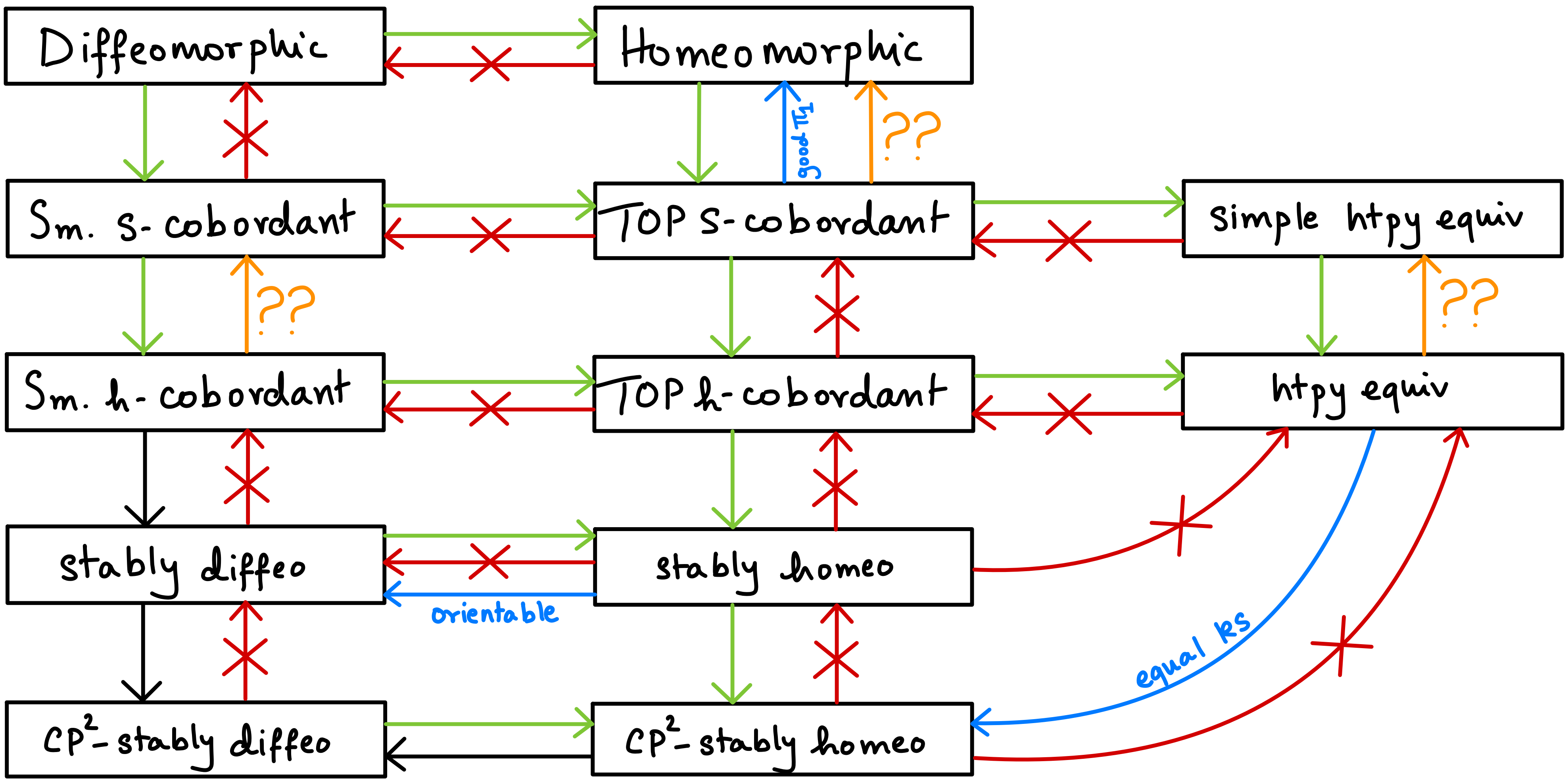
<u>Theorem [Kasprowski-Powell-R.]</u> $\forall\, n \geq 1$ $\exists\, \{N_i\}_{i=1}^{n}$ such that

N_i is closed, orientable, top 4-mfld

$\forall\, i \neq j$, N_i and N_j are
- simple homotopy equivalent
- top. h-cobordant
- not s-cobordant

$\pi_1 \cong \mathbb{Z}/2^r \times \mathbb{Z}$

Counterexamples in 4-manifold topology



Proof sketch

Let $M_r = L_{2^r,1} \times S^1$, $\quad \pi_r = \mathbb{Z}/2^r \times \mathbb{Z}$, $\quad n(r) = \left\lfloor \frac{2^{r-1}+4}{3} \right\rfloor + \left\lfloor \frac{r}{2} \right\rfloor - 1$

The surgery exact sequence(s):

$$\begin{tikzcd}
\mathcal{N}(M_r \times [0,1], \partial(M_r \times [0,1])) \arrow[r] \arrow[d] & L_5^s(\mathbb{Z}[\pi_r]) \arrow[r] \arrow[d] & \mathcal{S}^s(M_r) \arrow[d] \\
\mathcal{N}(M_r \times [0,1], \partial(M_r \times [0,1])) \arrow[r] & L_5^h(\mathbb{Z}[\pi_r]) \arrow[r] & \mathcal{S}^h(M_r)
\end{tikzcd}$$

Counterexamples in 4-manifold topology

Examples	Properties			Equivalence relations												
	smooth	oriented	$\pi_1 = 1$	equal χ	$S^2 \times S^2$-stably homeo.	$\mathbb{CP}^2$-stably homeo.	$S^2 \times S^2$-stably diffeo.	$\mathbb{CP}^2$-stably diffeo.	homotopy equiv.	simple homotopy equiv.	top. h-cobordant	top. s-cobordant	smoothly h-cobordant	smoothly s-cobordant	homeomorphic	diffeomorphic
S^4 and $S^2 \times S^2$	✓	✓	✓	✗	✓	✓	✓	✓	✗	✗	✗	✗	✗	✗	✗	✗
$S^2 \times S^2$ and $S^2 \widetilde{\times} S^2$	✓	✓	✓	✓	✗	✓	✗	✓	✗	✗	✗	✗	✗	✗	✗	✗
$\mathbb{CP}^2$ and $*\mathbb{CP}^2$	✗	✓	✓	✓	✗	✗	n/a	n/a	✓	✓	✗	✗	n/a	n/a	✗	n/a
$\mathbb{RP}^4 \# \mathbb{CP}^2$ and $\mathcal{R} \# *\mathbb{CP}^2$	✓	✗	✗	✓	✓	✓	✓	✓	✓	✓	✗	✗	✗	✗	✗	✗
$K3 \# \mathbb{RP}^4$ and $\#^{11} S^2 \times S^2 \# \mathbb{RP}^4$	✓	✗	✗	✓	✓	✓	✗	✓	✓	✓	✓	✓	✗	✗	✓	✗
$\mathbb{RP}^4$ and R	✓	✗	✗	✓	✓	✓	✗	✓	✓	✓	✓	✓	✗	✗	✓	✗
$L_{p,q_1} \times S^1, \dots, L_{p,q_k} \times S^1$, with $L_{p,q_1} \simeq L_{p,q_2}$ and $L_{p,q_1} \ncong L_{p,q_2}$	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✗	✗	✗	✗	✗	✗
$E(1)$ and $E(1)_{2,3}$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗
$\#^3 E_8$ and Le	✗	✓	✓	✓	✓	✓	n/a	n/a	✗	✗	✗	✗	n/a	n/a	✗	n/a
Kreck-Schafer manifolds	✓	✓	✗	✓	✓	✓	✓	✓	✗	✗	✗	✗	✗	✗	✗	✗
Teichner's $E \# E \# \#^k (S^2 \times S^2)$ and $*E \# *E \# \#^k (S^2 \times S^2)$	✓	✓	✗	✓	✗	✓	✗	✓	✓	✓	✗	✗	✗	✗	✗	✗
Akbulut's P and Q	✓	✗	✗	✓	✓	✓	✗	✓	✓	✓	✓	✓	✗	✗	✓	✗
$\mathcal{M}(L_{p,q} \times S^1)$, p odd, ∞ set	?	✓	✗	✓	✓	✓	n/a	n/a	✓	✓	✗	✗	n/a	n/a	✗	n/a
$\{M_r(\kappa)\}_{\kappa \in K}$	?	✓	✗	✓	✓	✓	n/a	n/a	✓	✓	✓	✗	n/a	n/a	✗	n/a

Questions?