Casson towers and filtrations of the smooth knot concordance group

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Doctoral defense
Rice University

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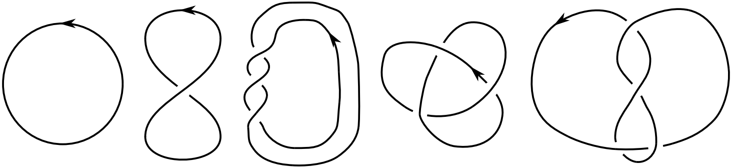
Knots



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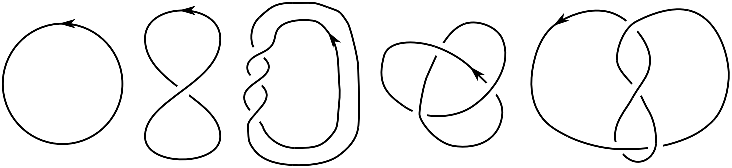

Take a piece of string, tie a knot in it, glue the two ends together.
A knot is a closed curve in space which does not intersect itself anywhere.

Equivalence of knots

Two knots are **equivalent** if we can get from one to the other by a continuous deformation, i.e. without having to cut the piece of string.

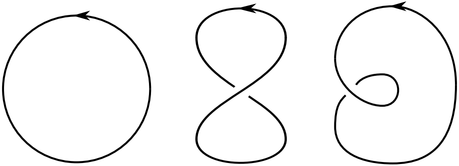

Figure: All of these pictures are of the same knot, the **unknot** or the **trivial knot**.

Knot theory is a subset of topology

Topology is the study of properties of spaces that are unchanged by continuous deformations.

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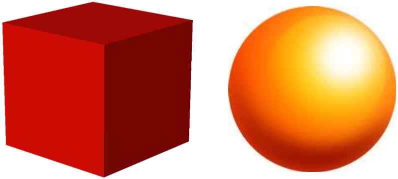

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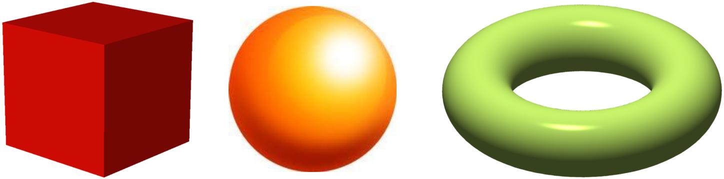

But a ball and a torus (doughnut) are different: we cannot continuously change a ball to a torus without tearing it in some way.

'Adding' two knots

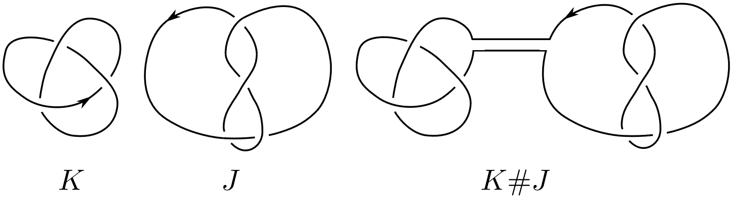

Figure: The connected sum operation on knots

The (class of the) unknot is the identity element, i.e. $K\#\text{Unknot} = K$.

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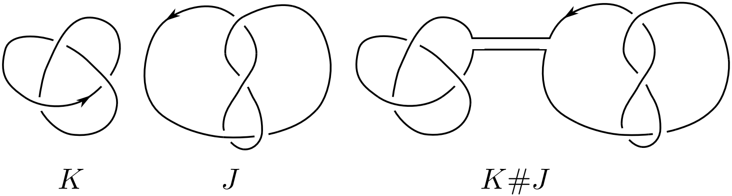

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The (class of the) unknot is the identity element, i.e. $K\#\text{Unknot} = K$.

However, there are no inverses for this operation. In particular, if neither K nor J is the unknot, then $K\#J$ cannot be the unknot either.

(In fact, we can show that $K\#J$ is more complex than K and J in a precise way.)

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A knot K is equivalent to the unknot **if and only if** it is the boundary of a disk.

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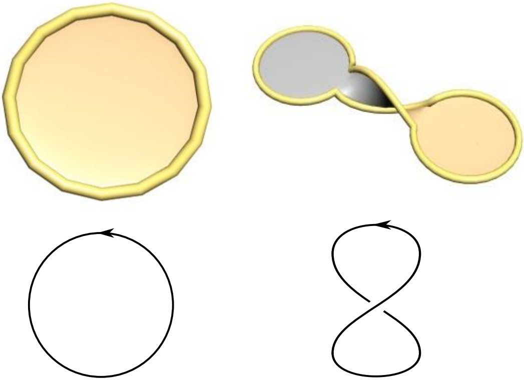

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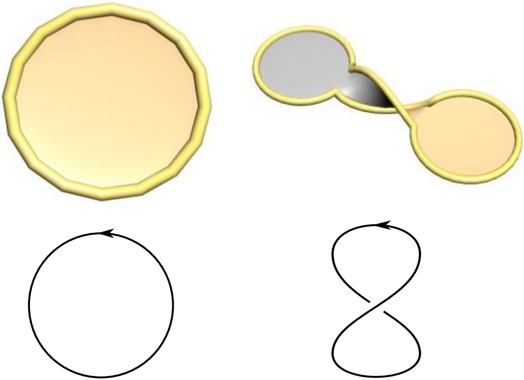

We want to extend this notion to four dimensions.

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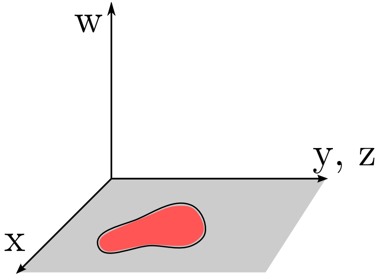

Figure: Schematic picture of the unknot

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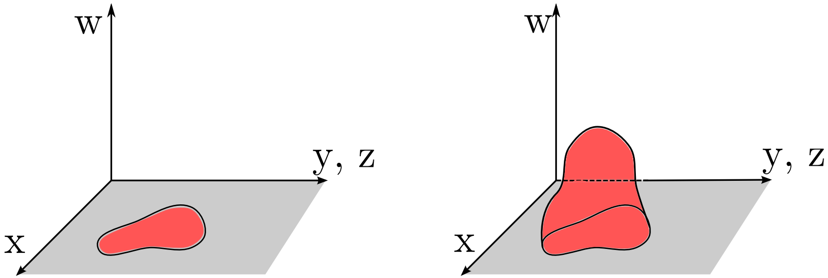

Figure: Schematic pictures of the unknot and a slice knot

Definition

A knot K is called **slice** if it bounds a disk in four dimensions as above.

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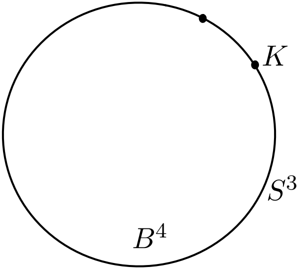

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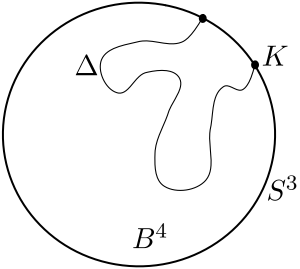

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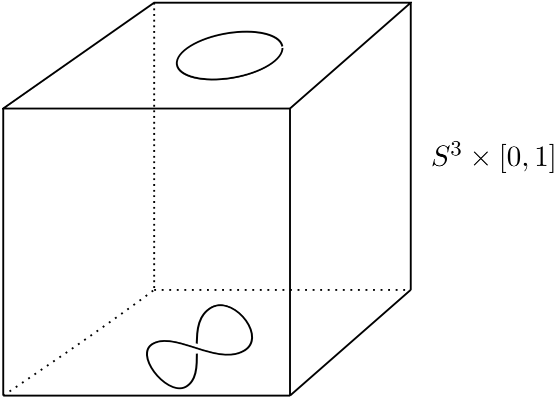
Knot concordance



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Two knots K and J are said to be **concordant** if they cobound a smooth annulus in $S^3 \times [0,1]$.

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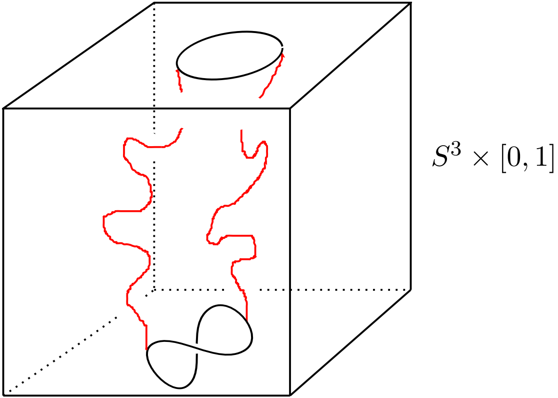

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Two knots K and J are said to be **concordant** if they cobound a smooth annulus in $S^3 \times [0, 1]$.

The knot concordance **group**

The set of knot concordance classes under the connected sum operation forms a group (i.e. for every knot K there is some $-K$, such that $K\# -K$ is a slice knot).

We call the group of knot concordance classes the (smooth) **knot concordance group** and denote it by $\mathcal{C}$.

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Similarly, we can define the *topological* knot concordance group, by only requiring a topological, locally flat embedding of an annulus.

There exist infinitely many smooth concordance classes of topologically slice knots (Endo, Gompf, etc.)

Why should we care about knots and knot concordance?

$$\frac{\text{Knots}}{\text{Isotopy}} \Longleftrightarrow \text{Classification of 3–manifolds}$$

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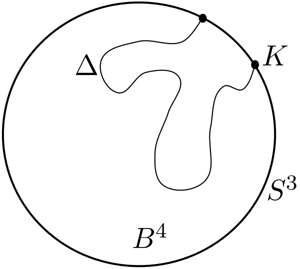
Approximating sliceness



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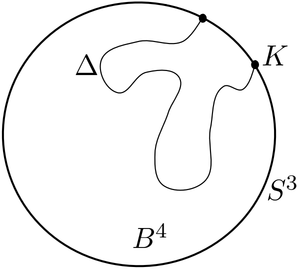

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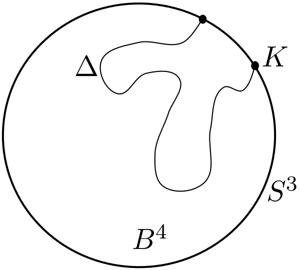

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The n–solvable filtration of $\mathcal{C}$

Definition (Cochran–Orr–Teichner, 2003)

For any $n \geq 0$, a knot K is in $\mathcal{F}_n$ (and is said to be n–*solvable*) if K bounds a smooth, embedded disk Δ in [[an approximation of B^4]].

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For any $n \geq 0$, a knot K is in $\mathcal{F}_n$ (and is said to be n–*solvable*) if K bounds a smooth, embedded disk Δ in a smooth, compact, oriented 4–manifold V with $\partial V = S^3$ such that

- $H_1(V) = 0$,
- there exist surfaces $\{L_1, D_1, L_2, D_2, \cdots, L_k, D_k\}$ embedded in $V - \Delta$ which generate $H_2(V)$ and with respect to which the intersection form is $\bigoplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,
- $\pi_1(L_i) \subseteq \pi_1(V - \Delta)^{(n)}$ for all i,
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Clearly,

$$\cdots \subseteq \mathcal{F}_n \subseteq \mathcal{F}_{n-1} \subseteq \cdots \subseteq \mathcal{F}_0 \subseteq \mathcal{C}$$

The n–solvable filtration of $\mathcal{C}$

- $\mathcal{F}_0 = \{K \mid \mathrm{Arf}(K) = 0\}$
- $\mathcal{F}_1 \subseteq \{K \mid K \text{ is algebraically slice}\}$
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- $\forall n$, $\mathbb{Z}^\infty \subseteq \mathcal{F}_n/\mathcal{F}_{n+1}$ (Cochran–Orr–Teichner, Cochran–Teichner, Cochran–Harvey–Leidy)

The grope filtration of $\mathcal{C}$

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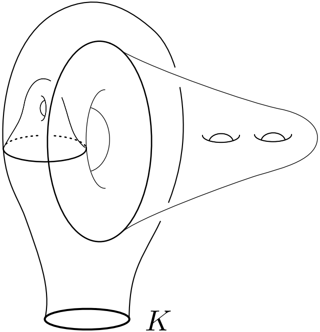

Figure: A grope of height 2

The grope filtration of $\mathcal{C}$

Model Theorem (Cochran–Orr–Teichner, 2003)

For all $n \geq 0$,

$$\mathcal{G}_{n+2} \subseteq \mathcal{F}_n$$

Topologically slice knots

Let $\mathcal{T}$ denote the set of all topologically slice knots.

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How can we use filtrations to study smooth concordance classes of topologically slice knots?

Positive and negative filtrations of $\mathcal{C}$

Definition (Cochran–Harvey–Horn, 2012)

For any $n \geq 0$, a knot K is in $\mathcal{P}_n$ (and is said to be n–*positive*) if K bounds a smooth, embedded disk Δ in [[an approximation of B^4]].

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These filtrations can be used to distinguish smooth concordance classes of topologically slice knots.

Goal

Model Theorem (Cochran–Orr–Teichner, 2003)

For all $n \geq 0$,

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Goal: Prove a version of the model theorem for the positive/negative filtrations.

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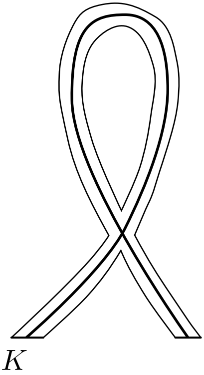
A **Casson tower** is built using layers of kinky disks, so they are natural objects to study in this context.

Casson towers



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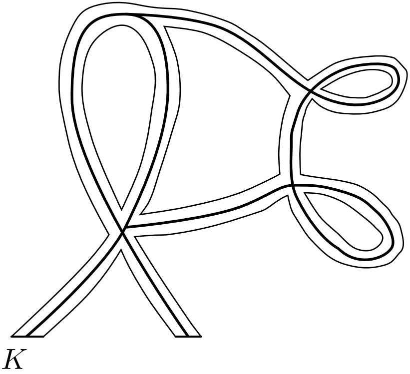

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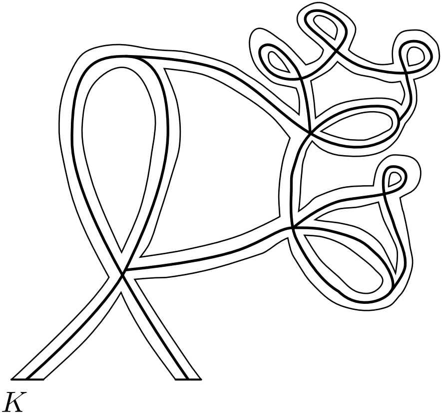

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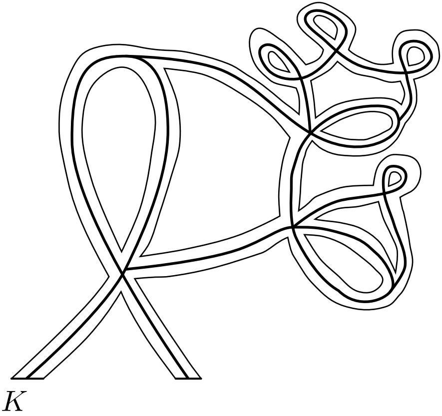

A Casson tower of height n consists of n layers of kinky disks.

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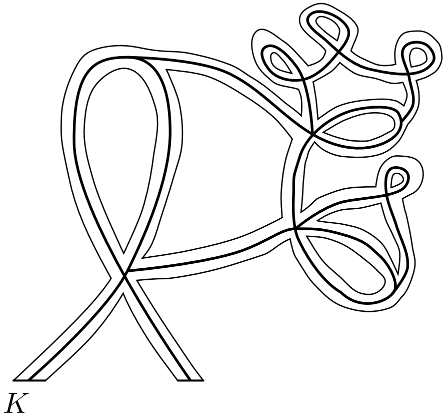

A Casson tower of height n consists of n layers of kinky disks.

A Casson tower T is of height $(2, n)$ if it has two layers of kinky disks, and each member of a standard set of generators of $\pi_1(T)$ is in $\pi_1(B^4 - T)^{(n)}$.

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This highly technical result led to a wealth of results about topological 4–manifolds, including the topological h–cobordism theorem in 4 dimensions (which implies the 4-dimensional topological Poincaré Conjecture) and Freedman's complete classification of topological 4–manifolds.

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- A knot is in $\mathfrak{C}_{2,\,n}$ if it bounds a Casson tower of height $(2,\,n)$ in B^4
- A knot is in $\mathfrak{C}^{+}_{2,\,n}$ if it bounds a Casson tower of height $(2,\,n)$ in B^4 such that all the kinks at the initial disk are positive
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Results

Model Theorem (Cochran–Orr–Teichner, 2003)

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- $\mathfrak{C}_{2,\,n} \subseteq \mathcal{F}_n$

Results

Proposition (R.)

For m–component links, let $\mathfrak{C}_n(m)$, $\mathfrak{C}_{2,n}(m)$, $\mathcal{F}_n(m)$, $\mathcal{P}_n(m)$, and $\mathcal{N}_n(m)$ denote the Casson tower, n–solvable, n–positive and n–negative filtrations respectively. For all n and $m \geq 2^{n+2}$,

$$\mathbb{Z} \subseteq \mathcal{F}_n(m)/\mathfrak{C}_{n+2}(m) \qquad \mathbb{Z} \subseteq \mathcal{F}_n(m)/\mathfrak{C}_{2,n}(m)$$

$$\mathbb{Z} \subseteq \mathcal{P}_n(m)/\mathfrak{C}^+_{n+2}(m) \qquad \mathbb{Z} \subseteq \mathcal{P}_n(m)/\mathfrak{C}^+_{2,n}(m)$$

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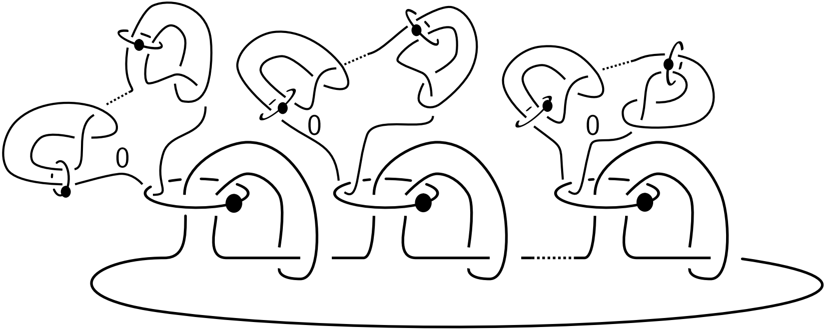

Figure: Kirby diagram for a general Casson tower of height two

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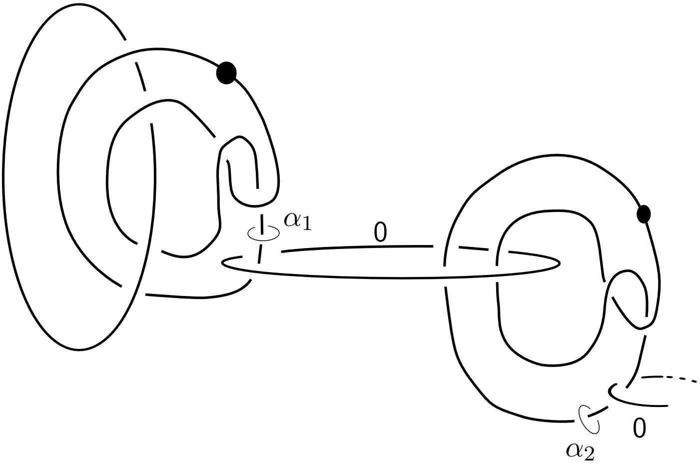

Figure: Kirby diagram for the first two stages of a simple Casson tower with a single positive kink at each stage

Thank you for your attention!