Satellite operations on knots, and fractals

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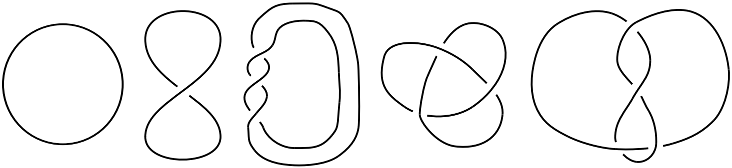
Knots



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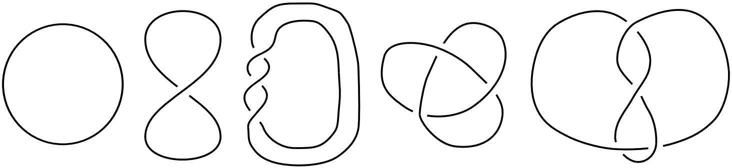

Take a piece of string, tie a knot in it, glue the two ends together.
A knot is a closed curve in space which does not intersect itself anywhere.

Equivalence of knots

Two knots are **equivalent** if we can get from one to the other by a continuous deformation, i.e. without having to cut the piece of string.

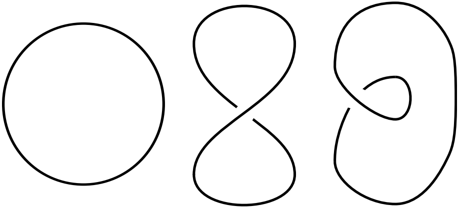

Figure : All of these pictures are of the same knot, the **unknot** or the **trivial knot**.

‘Adding’ two knots

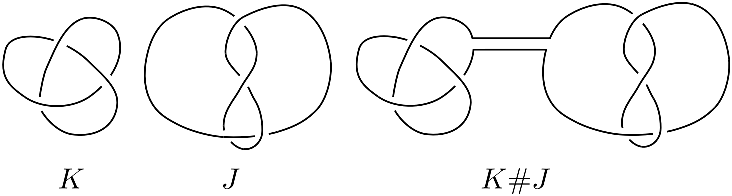

Figure : The connected sum operation on knots

The (class of the) unknot is the identity element, i.e. $K\#\text{Unknot} = K$

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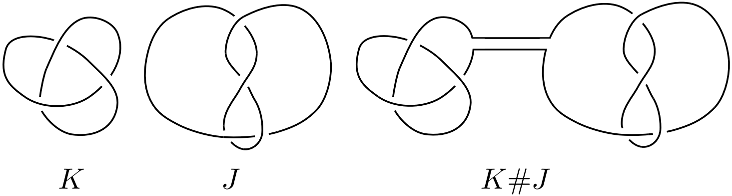

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However, there are no inverses for this operation. In particular, if neither K nor J is the unknot, then $K\#J$ cannot be the unknot either.

(In fact, we can show that $K\#J$ is more complex than K and J in a precise way.)

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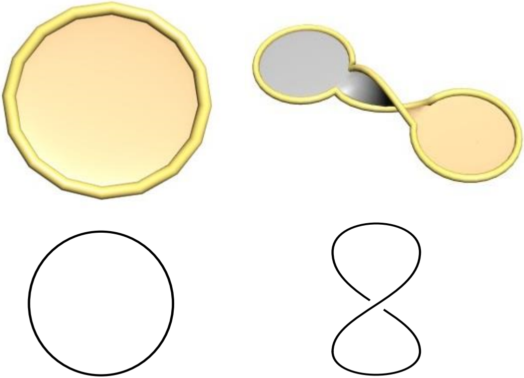

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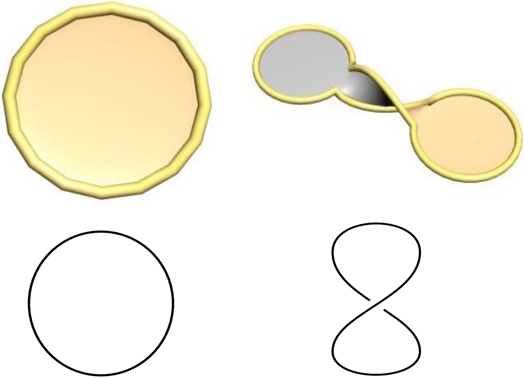


We want to extend this notion to four dimensions.

A 4–dimensional notion of a knot being ‘trivial’

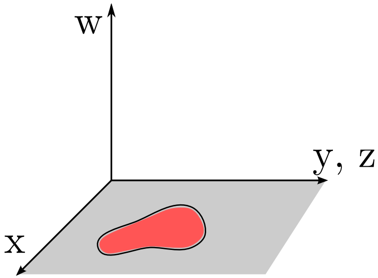


Figure : Schematic picture of the unknot

A 4–dimensional notion of a knot being 'trivial'

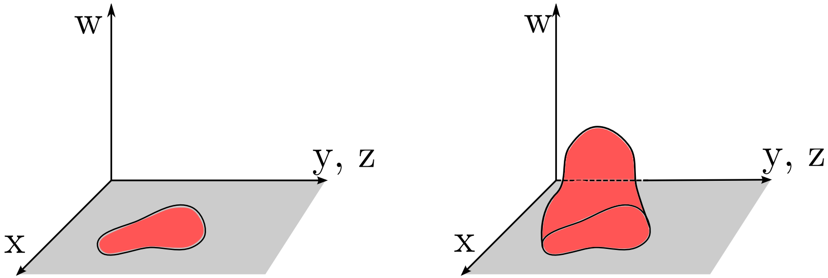
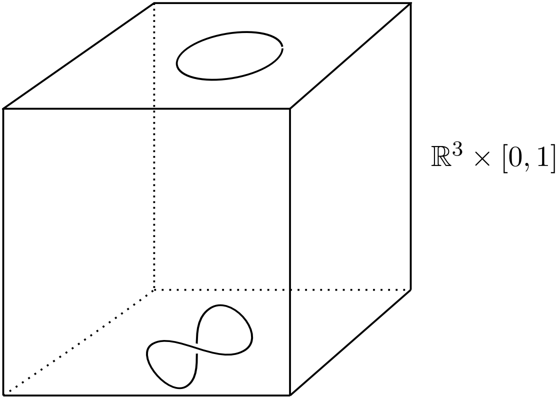


Figure : Schematic pictures of the unknot and a slice knot

Definition

A knot K is called **slice** if it bounds a disk in four dimensions as above.

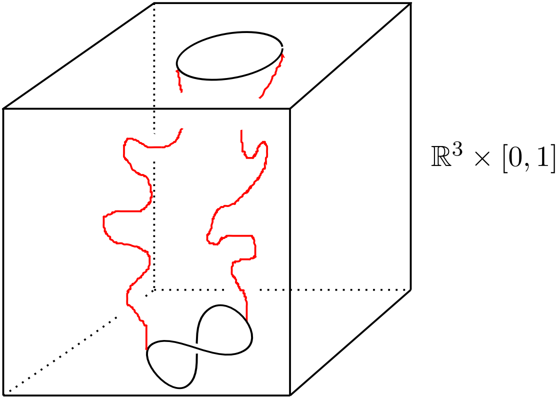
Knot concordance



Definition

Two knots K and J are said to be **concordant** if they cobound a smooth annulus in $\mathbb{R}^3 \times [0,1]$.

Knot concordance



Definition

Two knots K and J are said to be **concordant** if they cobound a smooth annulus in $\mathbb{R}^3 \times [0,1]$.

The knot concordance **group**

The set of knot concordance classes under the connected sum operation forms a group!

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This means that for every knot K there is some $-K$, such that $K\# -K$ is a slice knot.

We call the group of knot concordance classes the *knot concordance group* and denote it by $\mathcal{C}$.

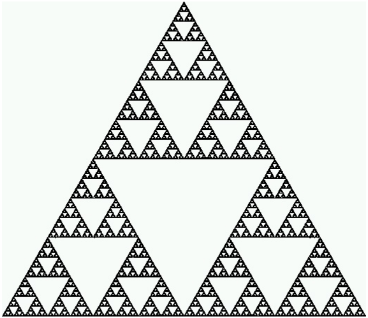
Goal

Goal: study the knot concordance group $\mathcal{C}$ by studying functions on it.

In particular, this will show that $\mathcal{C}$ has the structure of a fractal.

Fractals

Fractals are objects that exhibit ‘self-similarity’ at arbitrarily small scales.



i.e. there exist families of ‘injective’ functions from the set to smaller and smaller subsets.

Satellite operations on knots

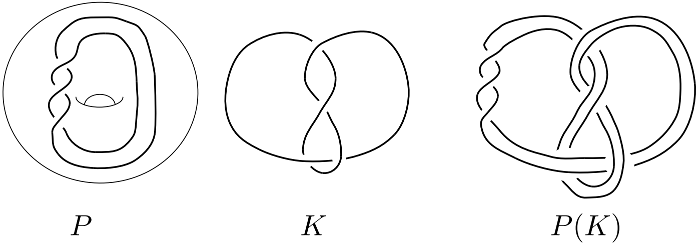


Figure : The satellite operation on knots

The satellite operation is a generalization of the connected sum operation.

Here P is called a satellite operator, and $P(K)$ is called a satellite knot.

Satellite operations on knots

Any knot P in a solid torus gives a function on the set of all knots

$$\begin{aligned} P : \mathcal{K} &\to \mathcal{K} \\ K &\to P(K) \end{aligned}$$

These functions descend to give well-defined functions on the knot concordance group.

$$\begin{aligned} P : \mathcal{C} &\to \mathcal{C} \\ K &\to P(K) \end{aligned}$$

The knot concordance group has fractal properties

Recall that a fractal is a set which admits self-similarities at arbitrarily small scales, i.e. there exist infinitely many injective functions from the set to smaller and smaller subsets.

Theorem (Cochran–Davis–R., 2012)

For large (infinite) classes of satellite operators P, $P : \mathcal{C} \to \mathcal{C}$ is injective (modulo the smooth 4–dimensional Poincaré Conjecture).

Theorem (R., 2013)

There exist infinitely many satellite operators P and a large class of knots K such that $P^i(K) \neq P^j(K)$ for all $i \neq j$.

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2. There is a four-dimensional equivalence relation on knots, called 'concordance', which gives the set of knots a group structure
3. By studying the action of 'satellite operators' on knots, we can see that the knot concordance group has fractal properties