A friendly introduction to knots in three and four dimensions

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SUNY Geneseo Mathematics Department Colloquium

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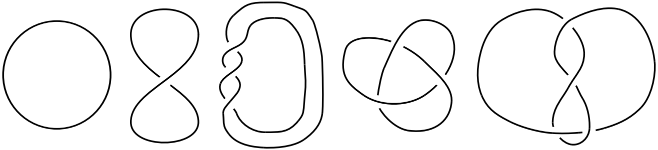
What is a knot?



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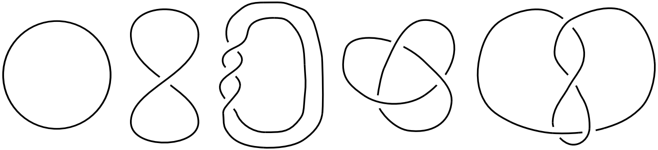
What is a knot?



Take a piece of string, tie a knot in it, glue the two ends together. A knot is a closed curve in space which does not intersect itself anywhere.

Equivalence of knots

Two knots are **equivalent** if we can get from one to the other by a continuous deformation, i.e. without having to cut the piece of string.

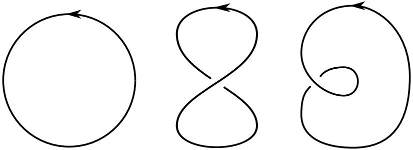


Figure: All of these pictures are of the same knot, the **unknot** or the **trivial knot**.

Knot theory is a subset of topology

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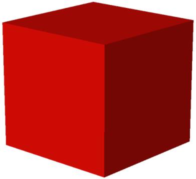

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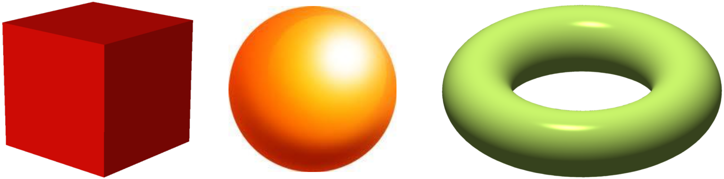

But a ball and a torus (doughnut) are different: we cannot continuously change a ball to a torus without tearing it in some way.

The historical origins of knot theory

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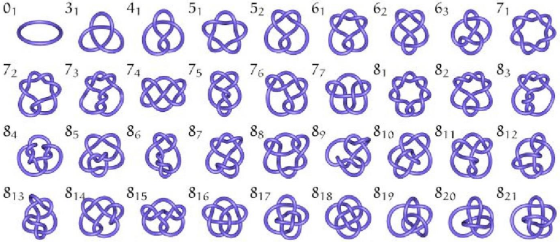

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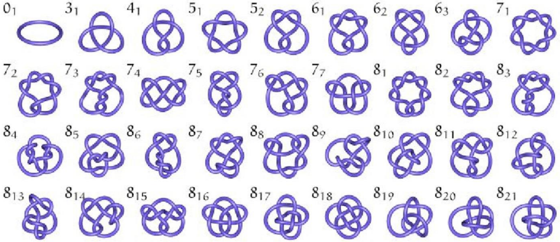

Tait thought he was making a periodic table! This view was held for about 20 years (until the Michelson–Morley experiment).

How can we tell if two knots are secretly the same?

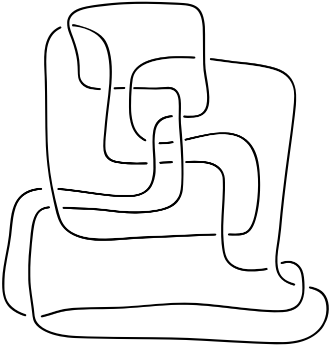

Figure: This is the unknot!

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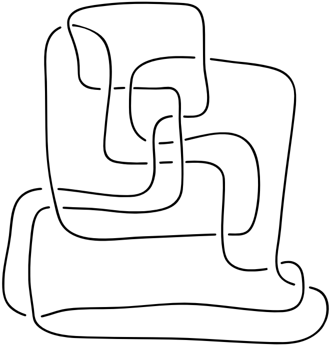

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How can we tell if knots are different? Is every knot secretly the unknot?

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(Does not help us figure out if two pictures are for the same knot)

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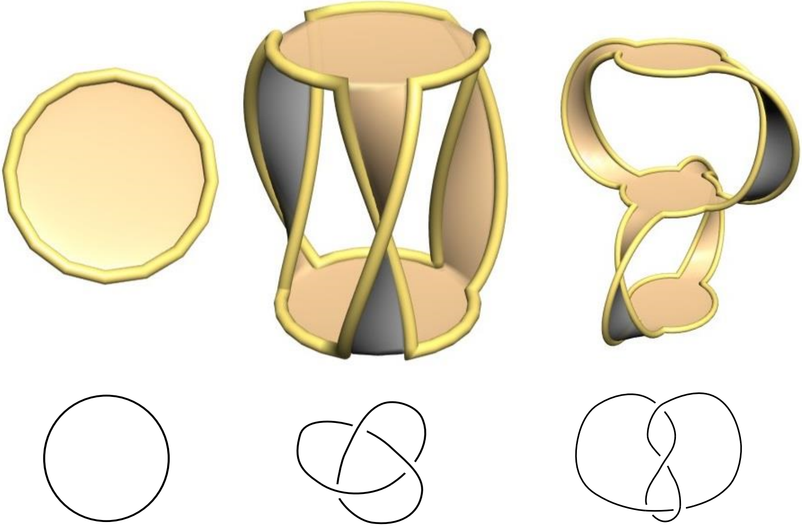

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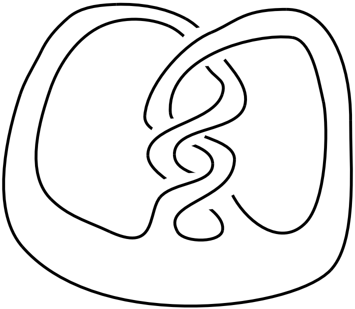

- Start with a knot

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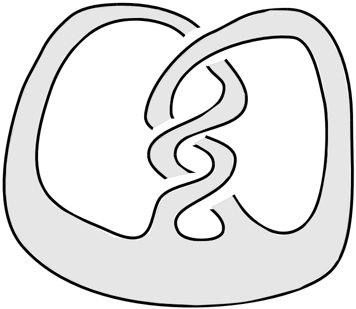

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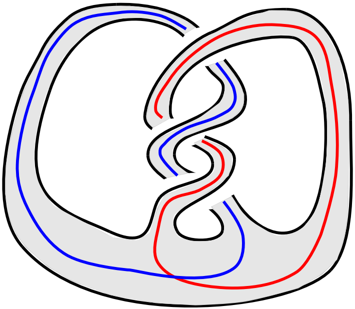
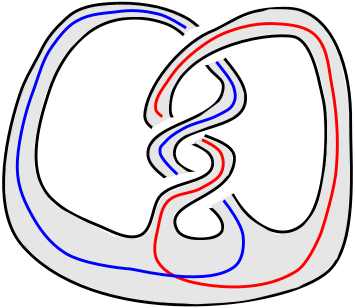


- Start with a knot
- Find a surface for it
- Find curves representing the 'spine' of the surface.

Example: Signature of a knot



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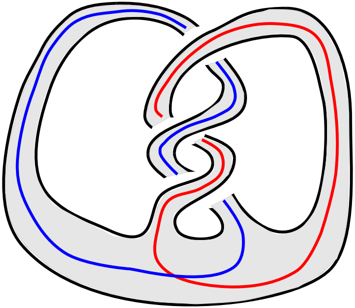

Using the **linking numbers** of these curves, we create a symmetric matrix: $V = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$;

$$M = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$$

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Definition

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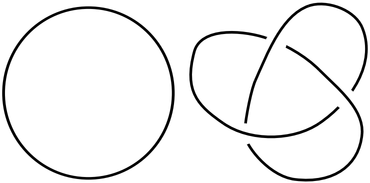

Figure: The unknot U and the trefoil T

$\sigma(U) = 0$ and $\sigma(T) = 2$
Therefore, the trefoil is not the trivial knot!

There exist infinitely many knots

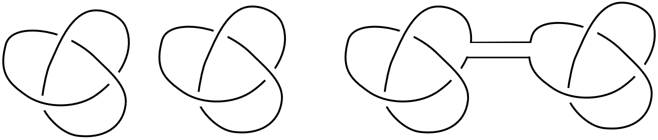

Figure: The connected sum of two trefoil knots, $T\#T$

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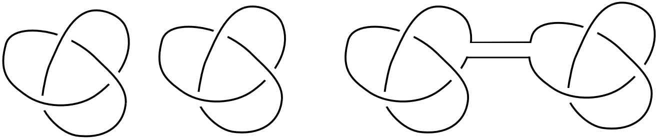

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As a result, there exist infinitely many knots!

Sample questions in knot theory in 3 dimensions

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1. Define knot invariants. We want invariants that are easy to compute, and which distinguish between large families of knots
2. Classify all knots
3. Is there an effective algorithm to decide if two knots are the same?
4. What is the structure of the set of all knots?

A 4–dimensional notion of a knot being ‘trivial’

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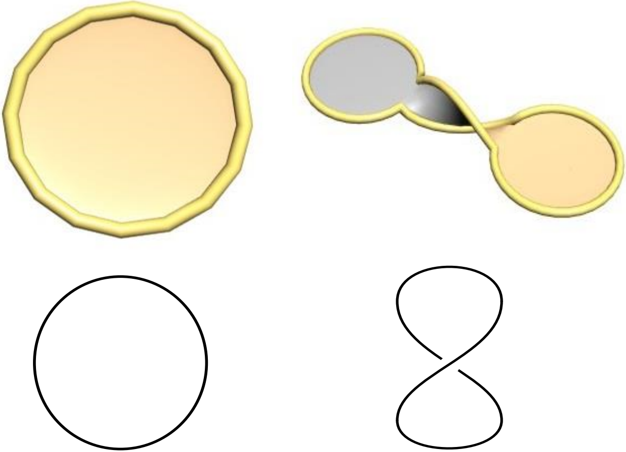

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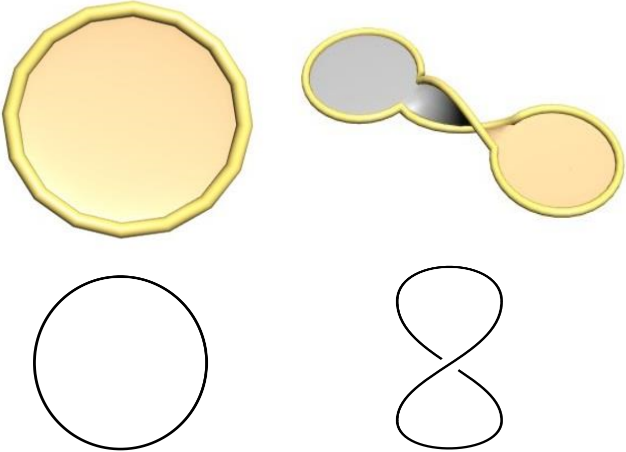

We want to extend this notion to 4 dimensions.

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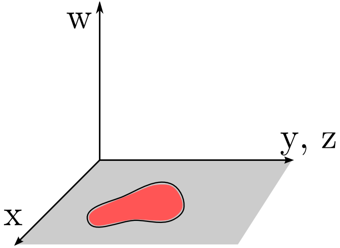

Figure: Schematic picture of the unknot

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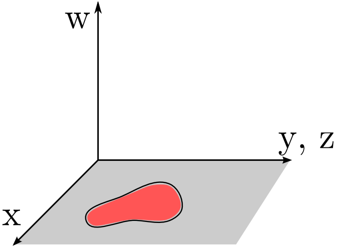
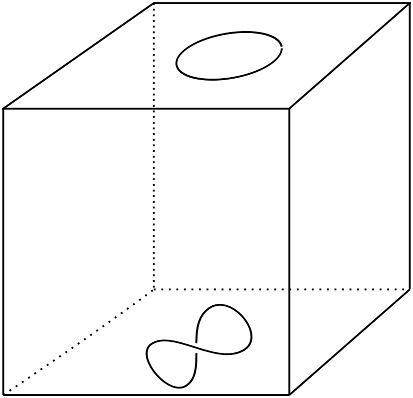


Figure: Schematic pictures of the unknot and a slice knot

Definition

A knot K is called **slice** if it bounds a disk in four dimensions as above.

Knot concordance



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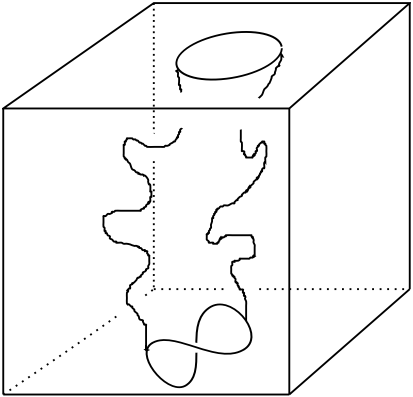

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Definition

Two knots K and J are said to be **concordant** if there is a cylinder between them in $\mathbb{R}^3 \times [0, 1]$.

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This means that for every knot K there is some $-K$, such that $K \# -K$ is a slice knot.

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Therefore, there are infinitely many non-slice knots!

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2. We can use **knot invariants** (such as signature) to determine when knots are distinct
3. There exist infinitely many knots
4. There is a 4–dimensional equivalence relation on the set of knots, called **concordance**
5. The set of concordance classes of knots forms a friendly algebraic object called a **group**