4–dimensional analogues of Dehn's lemma

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Classical Dehn's lemma in three dimensions

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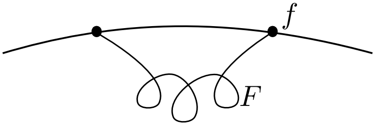

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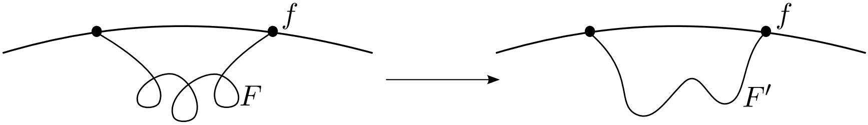

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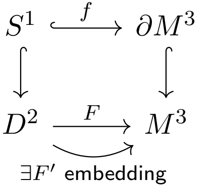

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1957: correct proof given by Papakyriakopoulos

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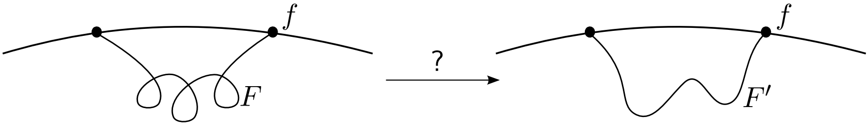
That is, if an embedded circle in the boundary of a 4–manifold is nullhomotopic in the interior, does it bound an embedded disk?



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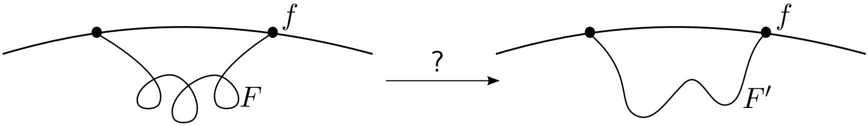

This is a question about *slice knots*, which are widely studied.

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Moreover, we can ask whether these embeddings exist *smoothly* or merely *topologically* (i.e. locally flat).

Results

Theorem (R.–Ruberman)

For embedded spheres/tori in the boundary of 4–manifolds, Dehn's lemma

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For embedded spheres/tori in the boundary of 4–manifolds, Dehn's lemma

1. *does not hold in general*
2. *holds under certain broad hypotheses*
3. *sometimes holds topologically but not smoothly*

Results for spheres

Theorem (R.–Ruberman)

There exists a sphere $S \subseteq \partial W^4$ where W is smooth and simply connected and S is nullhomotopic in W, but S does not bound a topological ball in W.

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Theorem (R.–Ruberman)

If $Y = Y_1 \#_S Y_2 = \partial W^4$ where Y_2 is an integer homology sphere, $\pi_1(W)$ is "good", and $\pi_1(Y_2) \to \pi_1(W)$ is the trivial map, then S bounds a topologically embedded ball in W.

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Any sphere $S \subseteq Y = \partial W^4$ where Y is an integer homology sphere and $\pi_1(W)$ is abelian bounds a topologically embedded ball in W.

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Theorem (R.–Ruberman)

There exists a sphere $S \subseteq Y = \partial W^4$ with W smooth and simply connected and Y an integer homology sphere such that S bounds a topologically embedded ball in W but no smooth ball in W.

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Example: Let P be the Poincaré homology sphere with a disk removed and γ a curve that normally generates $\pi_1(P)$. Let W be the 4–manifold obtained from $P \times [0, 1]$ by doing surgery along γ pushed into the interior. Then $\partial W = -P \# P$, where the connected-sum is performed along a sphere S.

Results for tori

Theorem (R.–Ruberman)

There exists an incompressible torus $T \subseteq Y = \partial W^4$ where W is contractible such that T extends to a map of the solid torus to W, but does not bound an embedded solid torus in W.

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Proposition (R.–Ruberman)

Let $T \subseteq Y = \partial W$ be a separating torus, $\gamma \subseteq T$ a simple closed curve, and e the surface induced framing. If

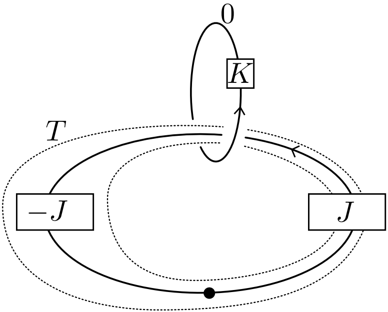
1. *γ is non-trivial in $H_1(T)$,*
2. *γ is smoothly (resp. topologically) slice in W with respect to e, and*
3. *the surgered manifold $Y_e(\gamma)$ is irreducible,*

then T bounds a smoothly (resp. topologically) embedded solid torus in W.

Results for tori

Theorem (R.–Ruberman)

There exists a contractible W and an incompressible torus $T \subseteq Y = \partial W$ such that T extends to a topological embedding of a solid torus in W, but not a smooth embedding.



Here J is the right-handed trefoil and K is the positive untwisted Whitehead double of the right-handed trefoil.