The fractal nature of the knot concordance group

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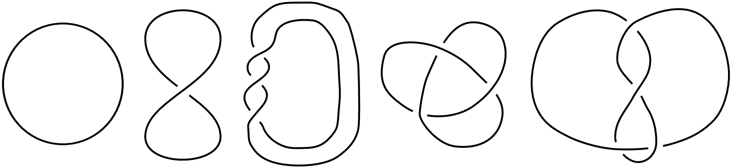
Knots



Definition

A knot is an embedding $S^1 \hookrightarrow S^3$, considered up to isotopy.

The set of knots is a monoid

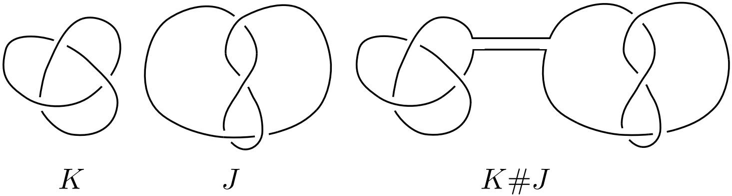
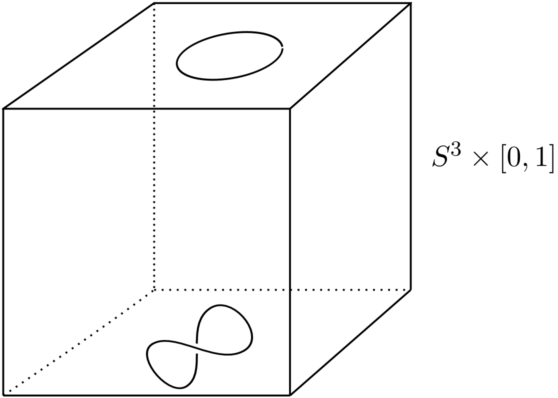


Figure : The connected sum operation on knots

The (isotopy class of the) unknot is the identity element.

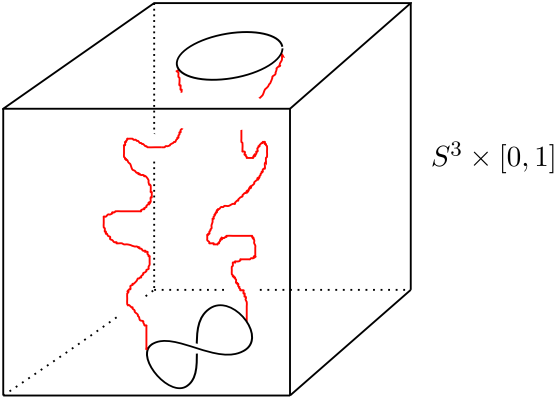
Knot concordance



Definition

Two knots K and J are said to be **concordant** if they cobound a smooth annulus in $S^3 \times [0,1]$.

Knot concordance



Definition

Two knots K and J are said to be **concordant** if they cobound a smooth annulus in $S^3 \times [0,1]$.

The knot concordance **group**

The set of knot concordance classes under the connected sum operation forms an abelian group!

This group is called the (smooth) knot concordance group, and is denoted by $\mathcal{C}$.

Why knots?

$$\frac{\text{Knots}}{\text{Isotopy}} \Longleftrightarrow \text{Classification of 3–manifolds}$$

$$\frac{\text{Knots}}{\text{Concordance}} \Longleftrightarrow \text{Classification of 4–manifolds}$$

Goal

Goal: study the knot concordance group $\mathcal{C}$ by studying functions on it.

In particular, this will show that $\mathcal{C}$ has the structure of a fractal.

Satellite operations on knots

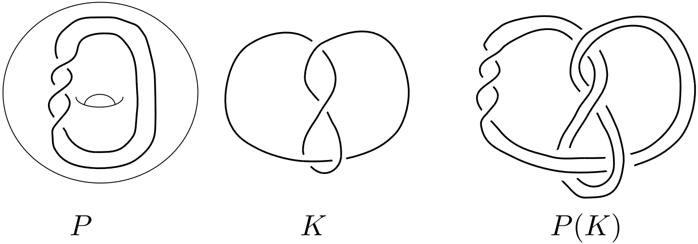


Figure : The satellite operation on knots

The satellite operation is a generalization of the connected sum operation.

Here P is called a satellite operator, and $P(K)$ is called a satellite knot.

Satellite operations on knots

Any knot P in a solid torus gives a function on the set of all knots

$$P : \mathcal{K} \to \mathcal{K}$$
$$K \to P(K)$$

These functions descend to give well-defined functions on the knot concordance group.

$$P : \mathcal{C} \to \mathcal{C}$$
$$K \to P(K)$$

The knot concordance group has fractal properties

A fractal is a set which admits self-similarities at arbitrarily small scales, i.e. there exist infinitely many injective functions from the set to smaller and smaller subsets.

Theorem (Cochran–Davis–R., 2012)

If P is a 'strong winding number one' satellite operator, then $P : \mathcal{C} \to \mathcal{C}$ is injective, modulo the smooth 4–dimensional Poincaré Conjecture.

Theorem (R., 2013)

There exist infinitely many 'strong winding number one' satellite operators P and a large class of knots K such that $P^i(K) \neq P^j(K)$ for all $i \neq j$.