Knots, four dimensions, and fractals

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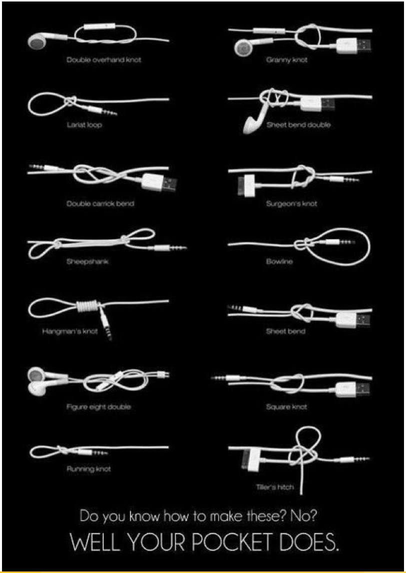
February 2, 2016

Examples of knots



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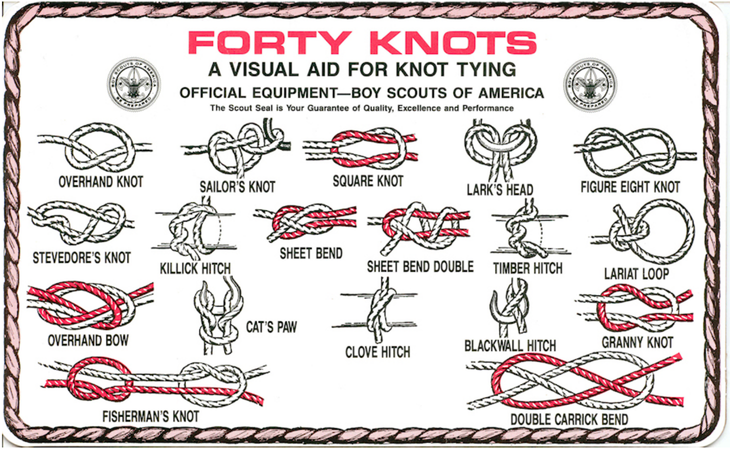
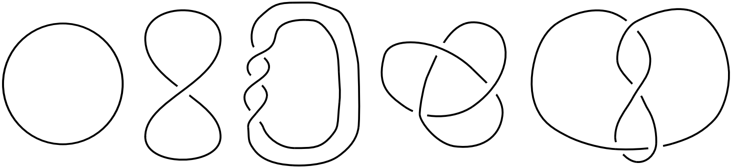


Mathematical knots



Take a piece of string, tie a knot in it, glue the two ends together.

Definition

A (mathematical) knot is a closed curve in space with no self-intersections.

Why knots?

Knot theory is a subset of the field of topology.

Theorem (Lickorish–Wallace, 1960s)

Any 3–dimensional 'manifold' can be obtained from $\mathbb{R}^3$ by performing an operation called 'surgery' on a collection of knots.

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Modern knot theory has applications to algebraic geometry, statistical mechanics, DNA topology, quantum computing, ...

Big questions in knot theory

1. How can we tell if two knots are equivalent?

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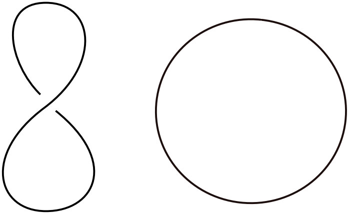

Figure: These are all pictures of the same knot!

Big questions in knot theory

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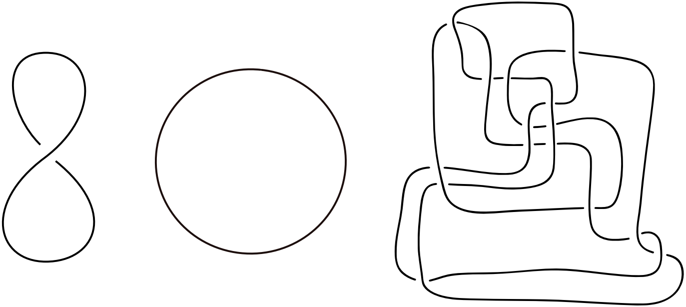

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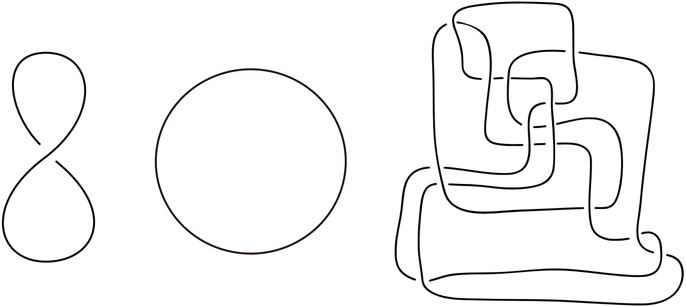

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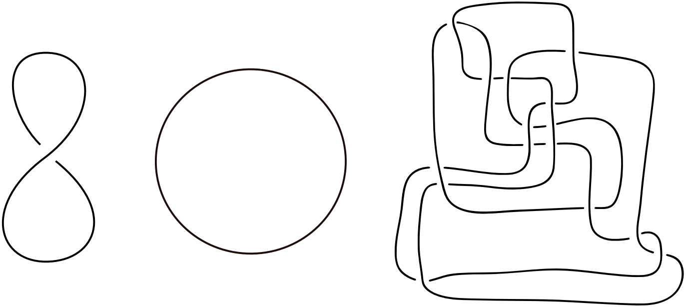


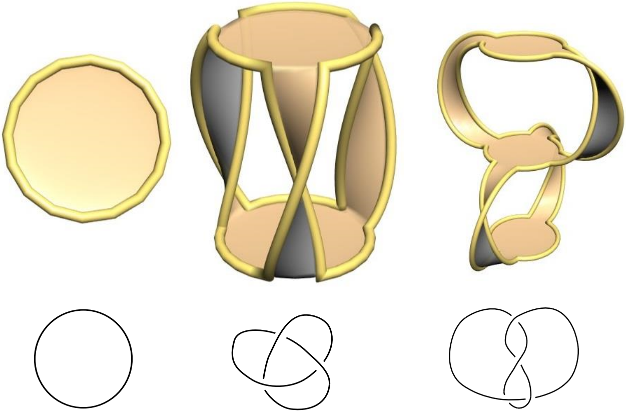
Figure: These are all pictures of the same knot!

2. How can we tell if two knots are distinct?
3. Can we quantify the 'knottedness' of a knot?

Genus of a knot

Proposition (Frankl–Pontrjagin, Seifert, 1930s)

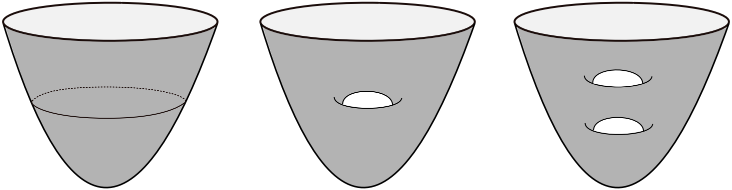
Any knot bounds a surface in $\mathbb{R}^3$.



Genus of a knot

Fundamental theorem in topology

Surfaces are classified by their genus.



genus$= 0$ genus$= 1$ genus$= 2$

Genus of a knot

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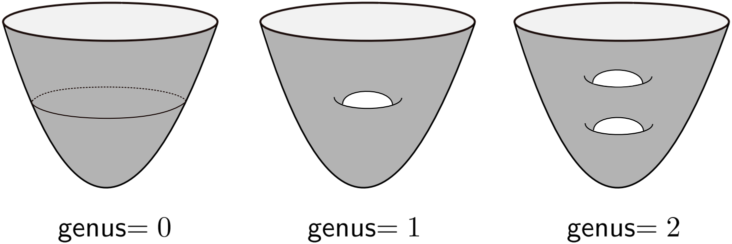

Definition

The *genus* of a knot K, denoted $\mathrm{g}(K)$, is the least genus of surfaces bounded by K.

Genus of a knot

Proposition

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If T is the trefoil knot, $g(T) = 1$. Therefore, the trefoil is not equivalent to the unknot.

Connected sum of knots

Figure: The connected sum of two trefoil knots, $T\#T$

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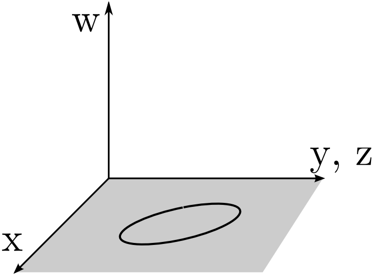
Corollary: We can never add together non-trivial knots to get a trivial knot.

Slice knots

Recall that a knot is equivalent to the unknot if and only if it is the boundary of a disk in $\mathbb{R}^3$.

Definition

A knot K is *slice* if it is the boundary of a disk in $\mathbb{R}^3 \times [0, \infty)$.



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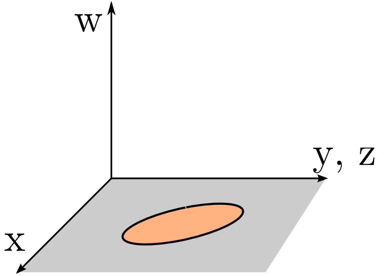

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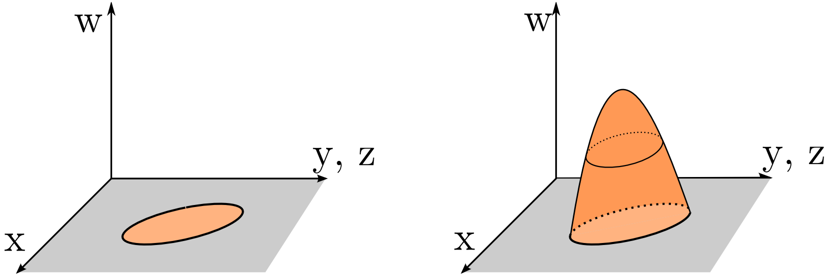
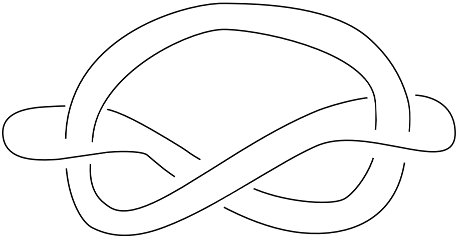


Figure: Schematic picture of the unknot and a slice knot

Examples of slice knots



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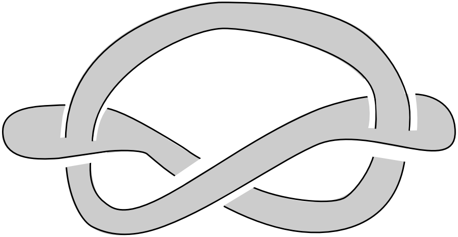

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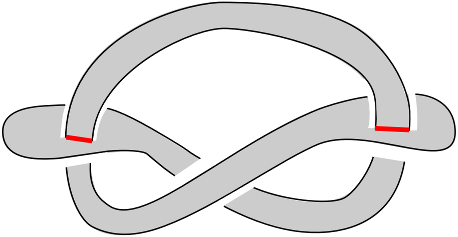

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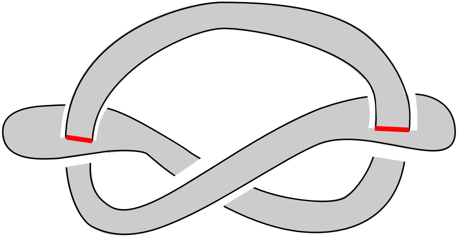

Knots of this form are called *ribbon knots*.

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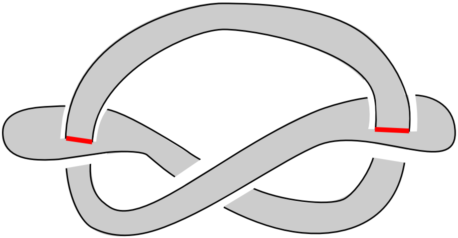

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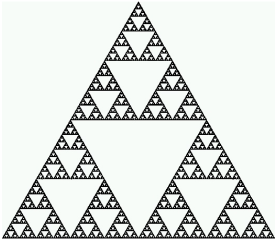
Knots, modulo slice knots, form a group called the *knot concordance group*, denoted $\mathcal{C}$.

Fractals

Fractals are objects that exhibit 'self-similarity' at arbitrarily small scales.



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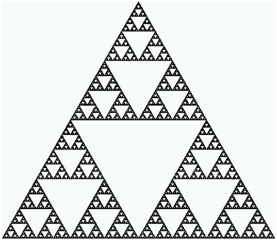

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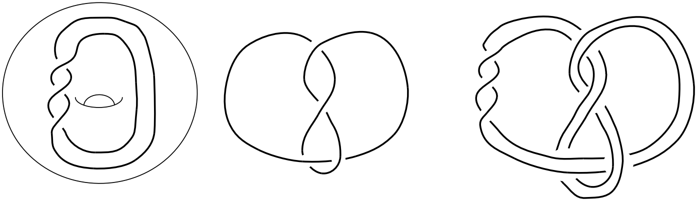
Conjecture (Cochran–Harvey–Leidy, 2011)

The knot concordance group $\mathcal{C}$ is a fractal.

Satellite operations on knots



P K $P(K)$

Figure: The satellite operation on knots

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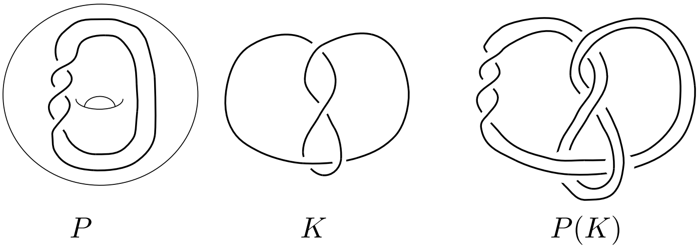

Figure: The satellite operation on knots

Any knot P in a solid torus gives a function on the knot concordance group,

$$P : \mathcal{C} \to \mathcal{C}$$
$$K \mapsto P(K)$$

These functions are called *satellite operators*.

The knot concordance group has fractal properties

Theorem (Cochran–Davis–R., 2012)

Large (infinite) classes of satellite operators $P : \mathcal{C} \to \mathcal{C}$ are injective.

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Theorem (A. Levine, 2014)

There exist satellite operators that are injective but not surjective.

Fractals

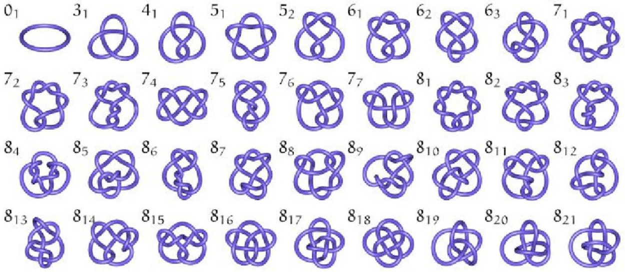
What is left to show?

In order for $\mathcal{C}$ to be a fractal, we need some notion of distance or size, to see that we have smaller and smaller embeddings of $\mathcal{C}$ within itself.

One way to do this is to exhibit a metric space structure on $\mathcal{C}$. There are several natural metrics on $\mathcal{C}$, but we have not yet found one that works well with the current results on satellite operators. The search is on!

The origins of mathematical knot theory

1880s: Kelvin (1824–1907) hypothesized that atoms were ‘knotted vortices’ in æther. This led Tait (1831–1901) to start tabulating knots.



Tait thought he was making a periodic table!

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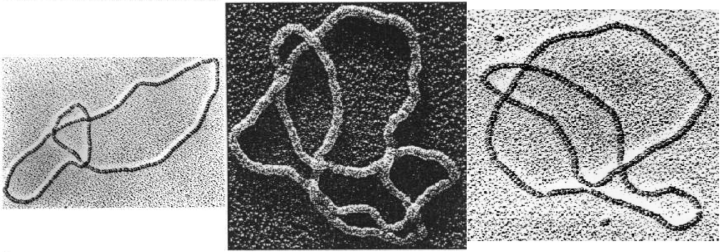

Figure: Knots in circular DNA.

(Images from Cozzarelli, Sumners, Cozzarelli, respectively.)