There exist infinitely many unknotted winding number one satellite operators on knot concordance

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Preliminaries

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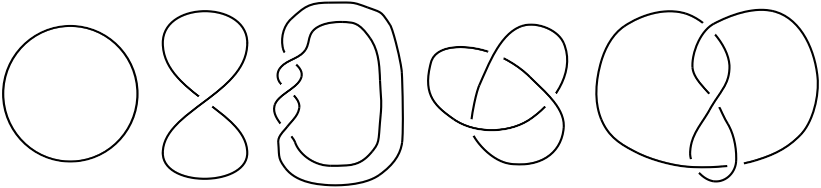
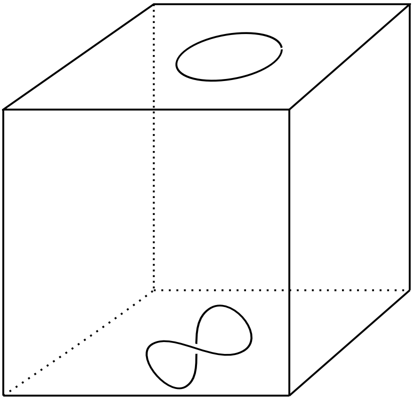


A 4–dimensional equivalence relation on knots



$S^3 \times [0, 1]$

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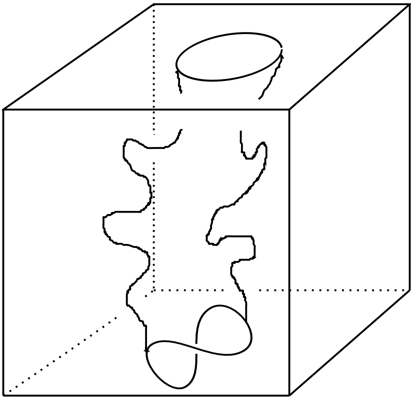

$S^3 \times [0, 1]$

Definition

Two knots K and J are said to be **concordant** if they cobound a a properly embedded smooth annulus in $S^3 \times [0, 1]$.

The knot concordance group

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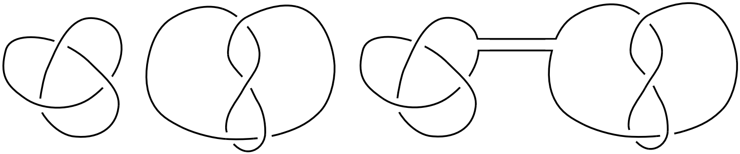
$$\text{Let } \mathcal{C} = \frac{\text{Knots}}{\text{concordance}}$$

$\mathcal{C}$ is a **group** under the connected-sum operation and is called the **knot concordance group**.



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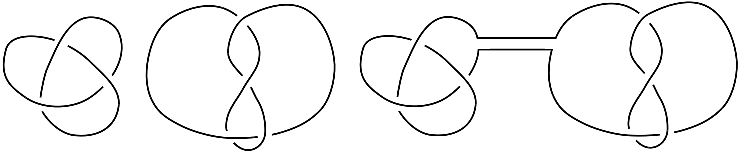

The identity element in $\mathcal{C}$ is the class of the unknot. That is, the class of knots which bound smoothly embedded disks in B^4, called **slice knots**.

Variants of the knot concordance group

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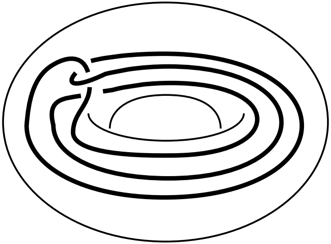
If the 4–dimensional (smooth) Poincaré Conjecture is true, $\mathcal{C} = \mathcal{C}^{\text{ex}}$.

The satellite construction

Definition

A **satellite operator**, or **pattern**, is a knot inside a solid torus, considered upto isotopy within the solid torus.



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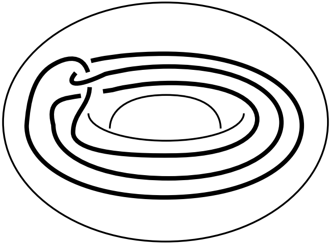
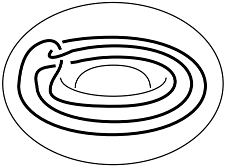


Definition

The **winding number** of a pattern is the signed count of its intersections with a meridional disk of the solid torus.

The satellite construction



P, the pattern

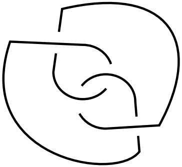


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Figure : The satellite operation on knots in S^3.

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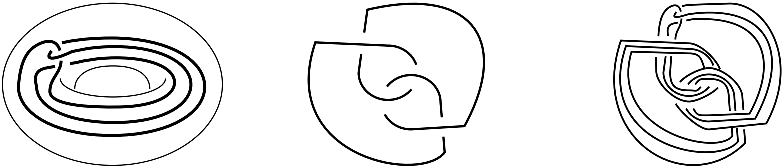

P, the pattern K, a knot in S^3 $P(K)$, the satellite knot

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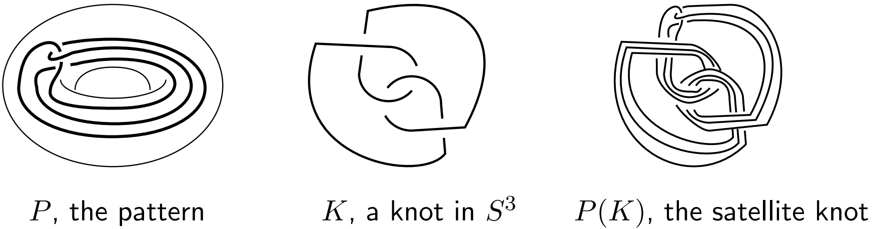

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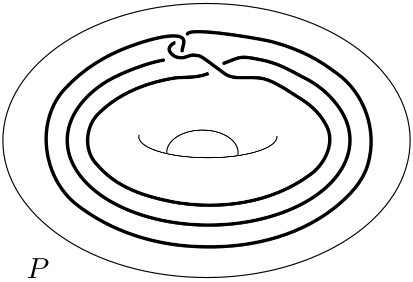
Remark

Any satellite operator P gives a function $P : \mathcal{C} \to \mathcal{C}$.

Strong winding number one operators



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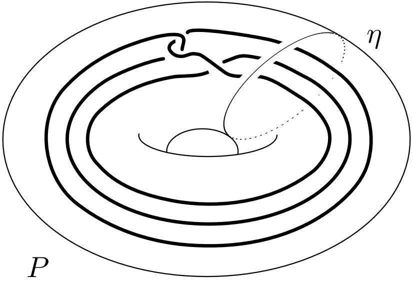

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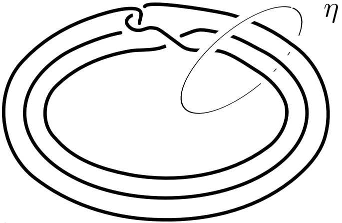

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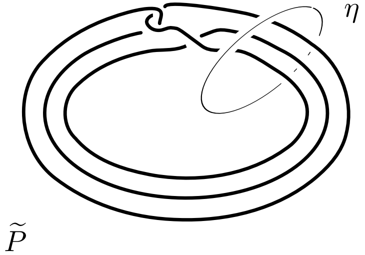

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If η, the meridian of the solid torus, normally generates $\pi_1(S^3 \backslash \widetilde{P})$, then P is said to have strong winding number one.

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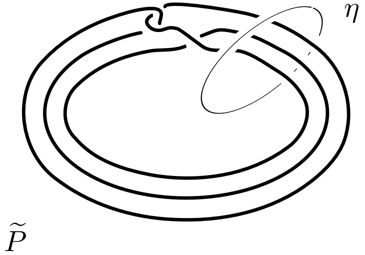

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For a P such that $\widetilde{P}$ is unknotted, P is strong winding number one if and only if it is winding number one.

Injectivity of satellite operators

Theorem (Cochran–Davis–R.,'12)

If P is a strong winding number one pattern, then

$$P : \mathcal{C}^{top} \to \mathcal{C}^{top} \text{ and } P : \mathcal{C}^{ex} \to \mathcal{C}^{ex}$$

are injective. That is, for any two knots K and J,

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If the 4–dimensional Poincaré Conjecture is true, $P : \mathcal{C} \to \mathcal{C}$ is injective.

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A fractal can be defined as a set which ‘exhibits self-similarity on many scales’.

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Question

How many strong winding number one operators are there?

Main theorem

Theorem (R.)

There is a strong winding number one satellite operator P and a large family of knots K such that $P^i(K) = P(P(\cdots(P(K))\cdots))$ are all distinct in $\mathcal{C}^{ex}$ and $\mathcal{C}$. That is, $P^i(K) \neq P^j(K)$ for all $i \neq j$.

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We can choose K to be topologically slice and $\widetilde{P}$ to be unknotted, in which case the set $\{P^i(K)\}$ is an infinite family of topologically slice knots that are distinct in smooth concordance.

τ–invariant of knots

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Ozsváth–Szabó defined the τ–invariant of a knot. This gives homomorphisms $\tau : \mathcal{C} \to \mathbb{Z}$ and $\tau : \mathcal{C}^{\text{ex}} \to \mathbb{Z}$.

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Proposition (Ozsváth–Szabó)

Start with a knot K_+. If K_- is the knot obtained by changing a single positive crossing of K_+, then

$$\tau(K_+) - 1 \leq \tau(K_-) \leq \tau(K_+)$$

Composition of patterns

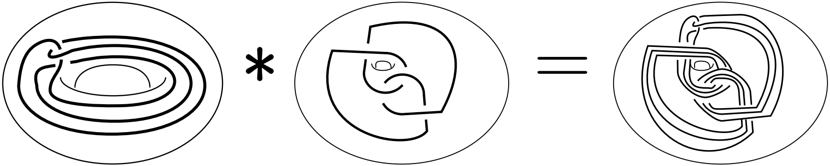

Figure : The monoid operation on patterns.

Fact

$$P(Q(K)) = (P * Q)(K)$$

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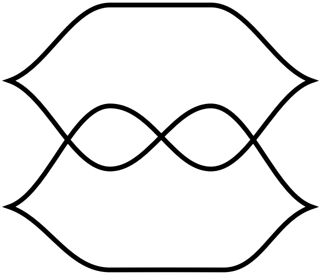
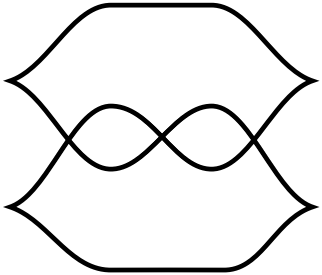


Classical invariants of Legendrian knots

$\text{tb}(K) = (\#\text{positive crossings } - \#\text{negative crossings}) - \frac{1}{2}\#\text{cusps}$
$\text{rot}(K) = \frac{1}{2}(\#\text{down cusps} - \#\text{up cusps})$



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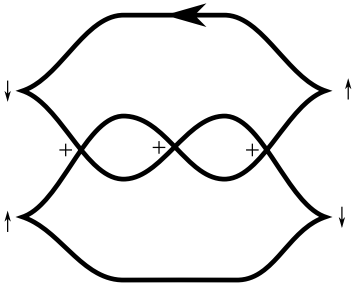
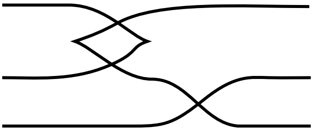


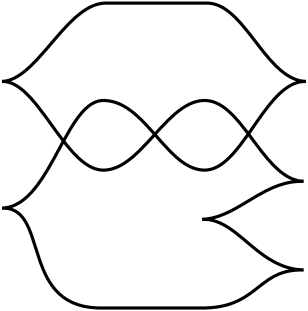
$\text{tb}(K) = (3-0) - \frac{1}{2}(4) = 1$, $\text{rot}(K) = \frac{1}{2}(2-2) = 0$

Classical invariants for Legendrian patterns



tb$(P) = 2$ and rot$(P) = 0$

The Legendrian satellite operation



For a knot K, suppose we have a Legendrian diagram with $\mathrm{tb}(K) = 0$.

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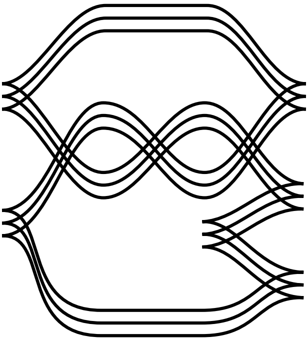

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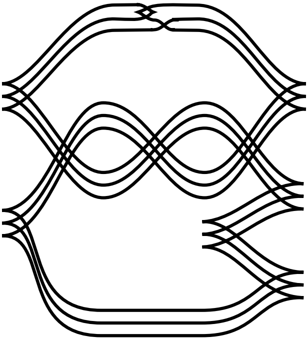

For a knot K, suppose we have a Legendrian diagram with $\text{tb}(K) = 0$. We can obtain the satellite knot $P(K)$ by taking parallels of K and then inserting the pattern.

Legendrian patterns and Legendrian satellites

Proposition (Ng)

$$tb(P(K)) = tb(P) + w(P)^2 tb(K)$$

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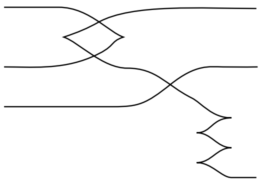
The slice–Bennequin inequality

Slice–Bennequin inequality (Rudolph)

For any knot K, we have that

$$\mathit{tb}(K) + |\mathit{rot}(K)| \leq 2\tau(K) - 1$$

Proof



$\text{tb}(P) = 0$ and $\text{rot}(P) = 2$

Proposition (Cochran–Franklin–Hedden–Horn)

For any knot K with $tb(K) = 0$, $rot(K) = 2\tau(K) - 1$ and $\tau(K) > 0$, $P(K) \neq K$ in $\mathcal{C}$ (and therefore, in $\mathcal{C}^{ex}$).

Note: There are large families of such knots K.

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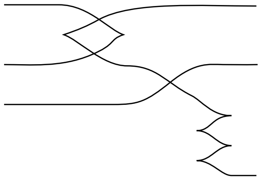

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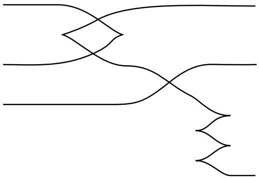

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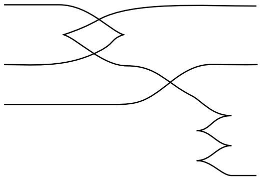

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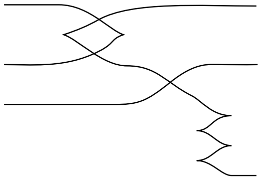

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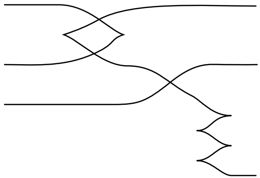

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$\Rightarrow P(K) \neq K$ □

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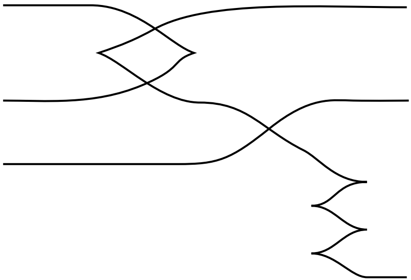

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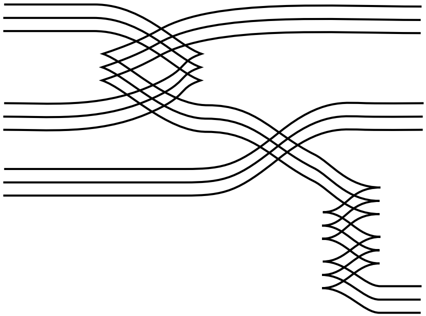

Proof

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$P^i(K) \neq K$ for any $i > 0$ in $\mathcal{C}$ (and therefore, in $\mathcal{C}^{ex}$).

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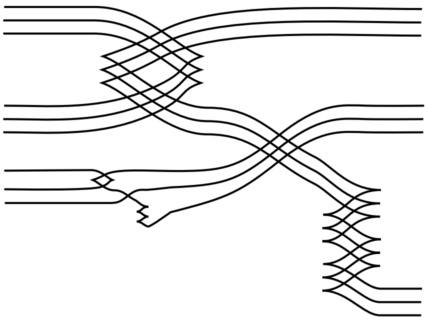

Figure : The operator P^2

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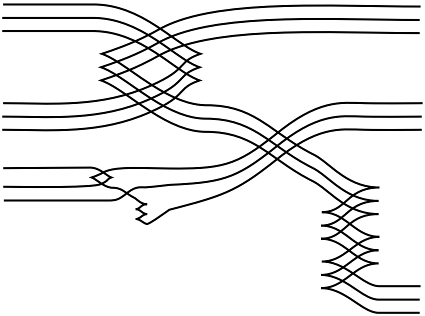

$\text{tb}(P^2) = \text{tb}(P) + \text{tb}(P)$

$\text{rot}(P^2) = \text{rot}(P) + \text{rot}(P)$

Figure : The operator P^2

Proof

$\mathrm{tb}(P^i) = 0$ and $\mathrm{rot}(P^i) = 2i$

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$$\mathsf{tb}(P^i(K)) = 0 \text{ and } \mathsf{rot}(P^i(K)) = 2\tau(K) - 1 + 2i$$

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$$\text{tb}(P^i(K)) + |\text{rot}(P^i(K))| \leq 2\tau(P^i(K)) - 1$$

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Therefore, $\tau(K) + i \leq \tau(P^i(K))$ and $P^i(K) \neq K$ for $i > 0$. □

Proof

Theorem (R.)

$P^i(K) \neq P^j(K)$ for any $i \neq j$ in $\mathcal{C}$ (and therefore, in $\mathcal{C}^{ex}$). Additionally, $\tau(P^i(K)) = \tau(K) + i$ for all $i \geq 0$

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Therefore,
$\tau(P^i(K)) \leq \tau(P^{i-1}(K)) + 1 \leq \tau(P^{i-2}(K)) + 2 \leq \cdots \leq \tau(K) + i.$

$$\Rightarrow \tau(P^i(K)) = \tau(K) + i \text{ for all } i > 0$$

□