A new family of links topologically, but not smoothly, concordant to the Hopf link

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Preliminaries

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Two links L_1 and L_2 are said to be **topologically concordant** if they cobound a disjoint collection of properly embedded locally flat annuli in $S^3 \times [0, 1]$.

Knot concordance groups

Smooth concordance classes of knots, under connected sum, form an abelian group called the **smooth knot concordance group**, denoted $\mathcal{C}$.

If we consider concordance in a potentially exotic copy of $S^3 \times I$, we still get an abelian group, called the **exotic knot concordance group**, denoted $\mathcal{C}^{\text{ex}}$.

Smooth vs. topological concordance

The differences between smooth and topological concordance model the differences between smooth and topological 4–manifolds, e.g. a knot which is topologically concordant to the unknot, but not smoothly concordant, gives rise to an exotic $\mathbb{R}^4$.

There exist infinitely many examples of knots that are topologically concordant to the unknot but not smoothly concordant.

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We give another infinite family of examples, using different techniques. We also show that our examples are distinct from the above.

Satellite knots

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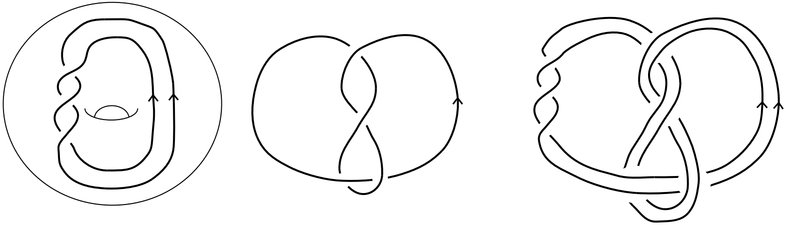
Any pattern acts on knots via the classical satellite construction.



P K $P(K)$

Satellite operators

The satellite construction descends to well-defined functions on $\mathcal{C}$ and $\mathcal{C}^{\text{ex}}$, called **satellite operators**, i.e. we get

$$\begin{aligned} P : \mathcal{C} &\to \mathcal{C} \\ K &\mapsto P(K) \end{aligned}$$

and

$$\begin{aligned} P : \mathcal{C}^{\text{ex}} &\to \mathcal{C}^{\text{ex}} \\ K &\mapsto P(K) \end{aligned}$$

Link concordance and satellite operators

Proposition (Cochran–Davis–R.)

If the 2–component links L_0 and L_1 with unknotted second component are concordant (or even exotically concordant), then the corresponding patterns P_0 and P_1 induce the same satellite operator on $\mathcal{C}^{ex}$, i.e. for any knot K, $P_0(K)$ and $P_1(K)$ are exotically concordant.

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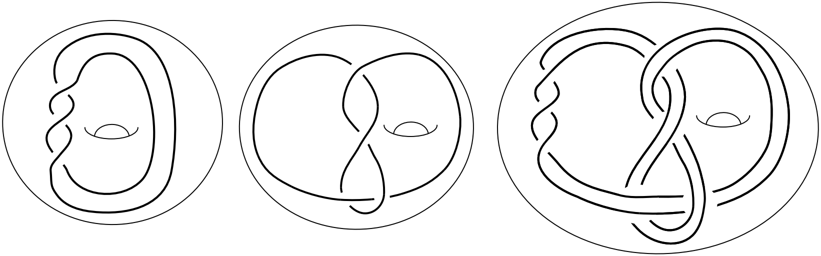
Notice that the Hopf link corresponds to the pattern consisting of the core of a solid torus, which induces the identity satellite operator.

This translates the question of whether 2–component links are concordant to a question of whether a satellite operator is distinct from the identity function.

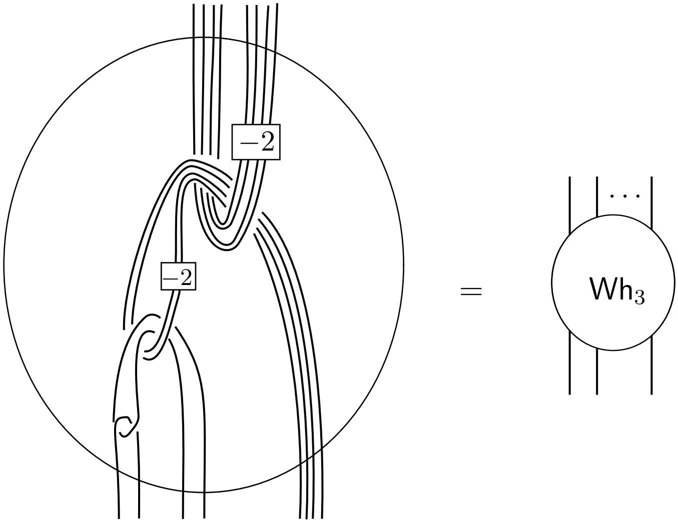
Iterated patterns

We can compose patterns as follows:



P Q $P \star Q$

Our links



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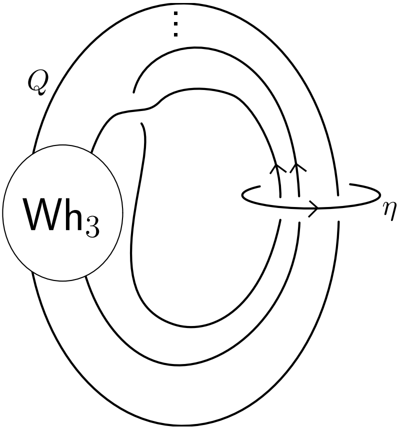


Let $L = (Q, \eta)$.

Our links

Theorem (Davis–R.)

The links $\{(Q^i, \eta(Q^i))\}$ *are each topologically concordant to the Hopf link, but are distinct from the Hopf link (and one another) in smooth concordance.*

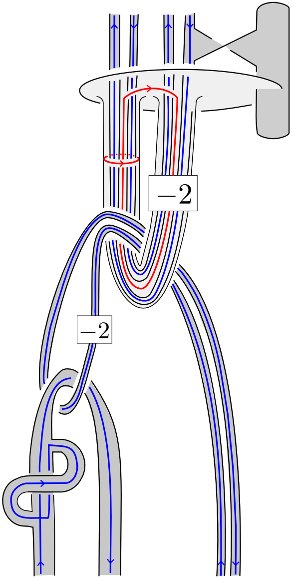
Topological concordance to Hopf

Start with $L = (Q, \eta)$.

Method 1: Use the fact that the link "Wh_3" is topologically slice (Freedman)

Method 2: Compute the Alexander polynomial using a C–complex.

Topological concordance to Hopf link

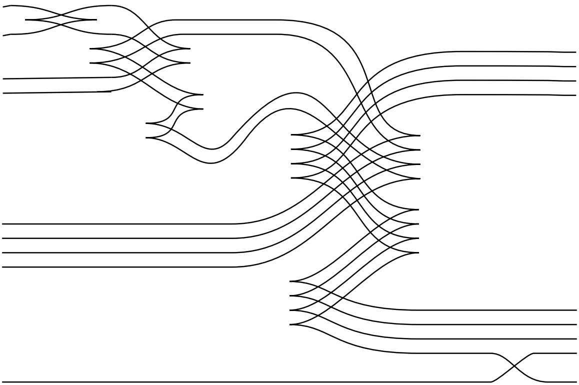


Topological concordance to Hopf link

We have that (Q, η) is topologically concordant to the Hopf link. We can modify the concordance by performing satellite operations on the annulus for the first component. This gives a topological concordance between (Q, η) and $(Q^2, \eta(Q^2))$. Iterate to see that each member of the family $\{(Q^i, \eta(Q^i))\}$ is topologically concordant to the Hopf link.

Distinctness in smooth concordance

We have a Legendrian diagram for the pattern Q.



$$\mathrm{tb}(\mathcal{Q}) = 2, \mathrm{rot}(\mathcal{Q}) = 0$$

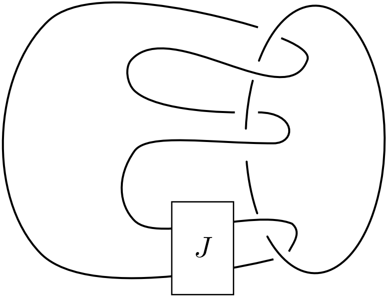
Distinctness in smooth concordance

Proposition (R.)

If P is a winding number one pattern such that $P(U)$ is unknotted, where U is the unknot, and P has a Legendrian diagram $\mathcal{P}$ with tb$(\mathcal{P}) > 0$ and tb$(\mathcal{P}) +$ rot$(\mathcal{P}) \geq 2$, then the iterated patterns P^i induce distinct functions on $\mathcal{C}^{ex}$, i.e. there exists a knot K such that $P^i(K)$ is not exotically concordant to $P^j(K)$, for each pair of distinct $i, j \geq 0$.

Here P^0 is the identity satellite operator, so in particular, the above shows that our links are not smoothly concordant to the Hopf link.

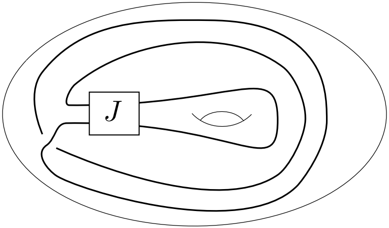
Our links are different from previous examples



Proposition (Davis–R.)

The links $\{(Q^i, \eta(Q^i)) \mid i \geq 4\}$ *are distinct from the links* ℓ_J *constructed by Cha–Kim–Ruberman–Strle.*

Previous examples



These are the patterns L_J corresponding to the previous examples.

We can compute that for RHT the right-handed trefoil,

$$-2 \leq \tau(L_J(RHT)) \leq 4.$$

In contrast, for our examples, $i + 1 \leq \tau(Q^i(RHT))$.