Moscow-Beijing topology seminar

June 18, 2025

Constructing locally flat surfaces in 4-manifolds

Locally flat surfaces in 4-manifolds

Talk 1:

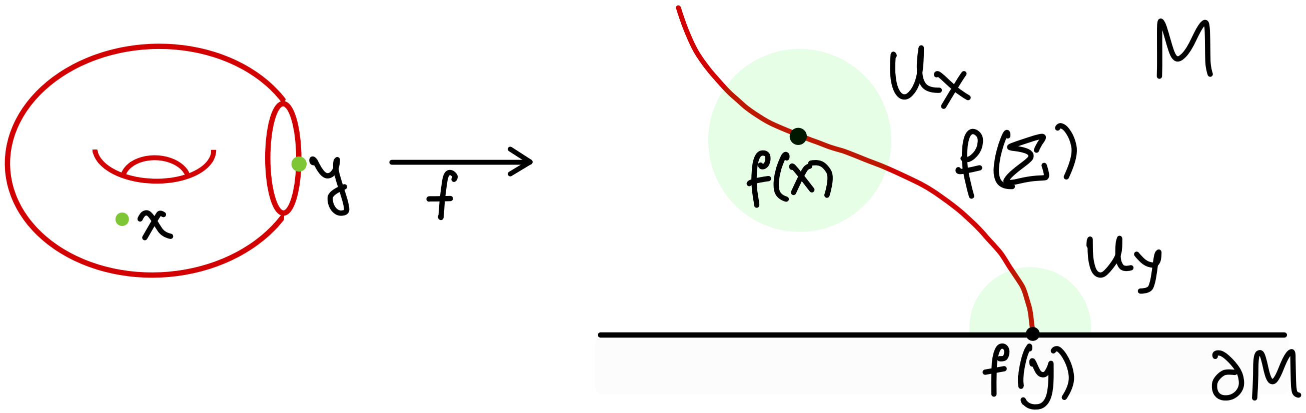
Theorem [Lee-Wilczyński]: Let M be a closed, simply connected 4-manifold.
Then every primitive $\alpha \in H_2(M;\mathbb{Z})$ is represented by a locally flat embedded torus.

Talk 2 (next week):

Theorem [Freedman-Quinn]: Let $K \subseteq S^3$ be a knot with Alexander polynomial one.
Then K is topologically slice.

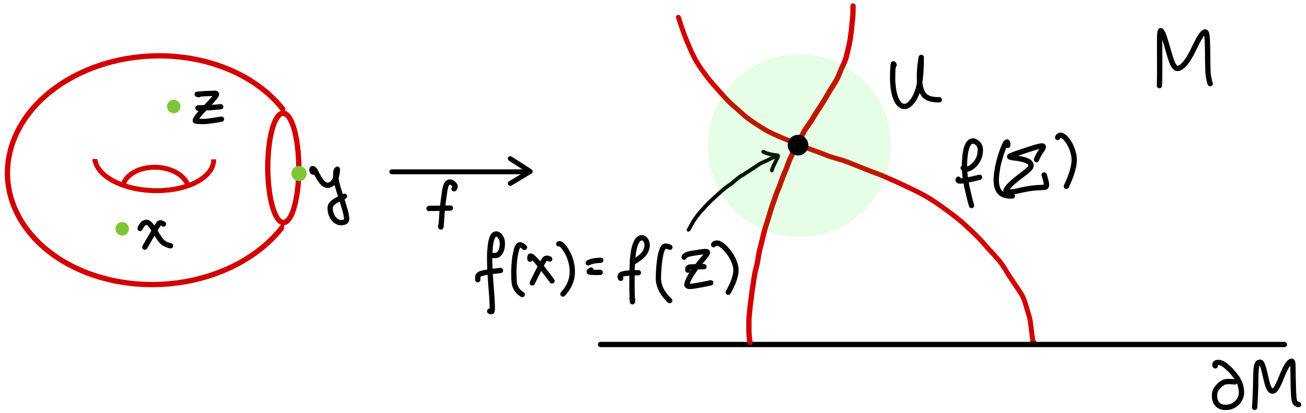
Locally flat surfaces in 4-manifolds

Definition: Let M be a 4-manifold and Σ a surface. An embedding $f: \Sigma \hookrightarrow M$ is said to be locally flat if $\forall\, x \in \Sigma$, there exists $U \ni f(x)$ with $(U, U \cap f(\Sigma)) \approx (\mathbb{R}^4, \mathbb{R}^2)$



Generic immersions

Definition: Let M be a 4-manifold and Σ a surface. A map $f:\Sigma \to M$ is said to be a *generic immersion* if it is a locally flat embedding except at isolated, transverse double point singularities, i.e. points with neighbourhoods U, such that $(U, U \cap f(\Sigma)) \approx (\mathbb{R}^4, \mathbb{R}^2_{xy} \cup \mathbb{R}^2_{zw})$



Then we write $f:\Sigma \looparrowright M$.

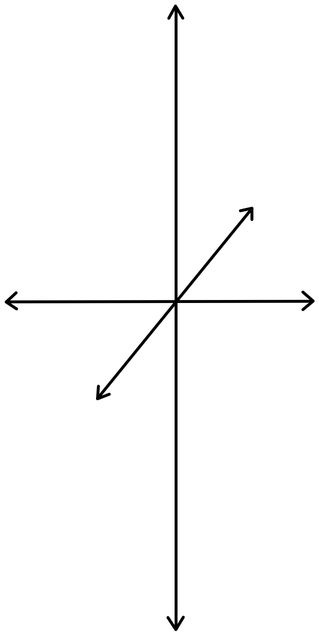
Fundamental tools

Theorem [Freedman-Quinn]: Every continuous map $\Sigma \to M$ is homotopic to a generic immersion.

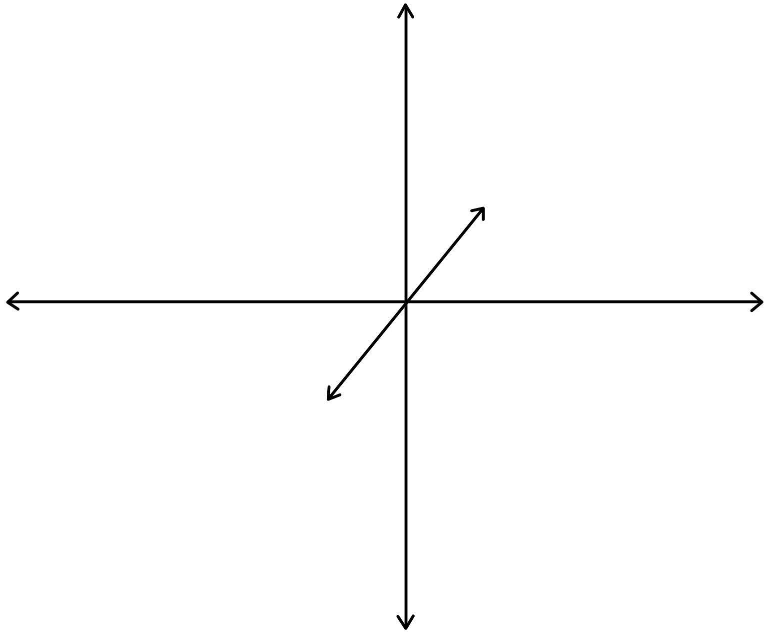
Theorem [Quinn]: Every locally flat submanifold of M has a (linear) normal bundle.

Theorem [Quinn]: Let Σ_1, Σ_2 denote locally flat submanifolds of M. Then there is an ambient isotopy of M taking Σ_1 to Σ_1' such that Σ_1' and Σ_2 intersect transversely.

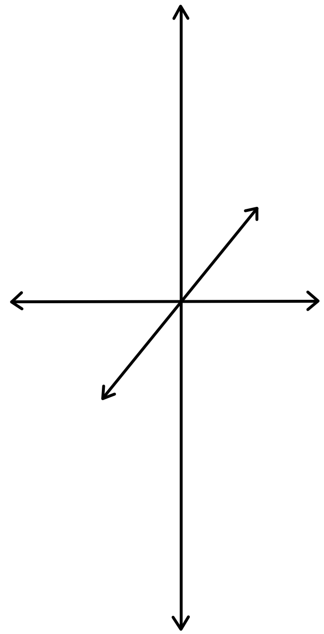
Visualising locally flat embeddings/generic immersions



$\mathbb{R}^3 \times -\varepsilon$



$\mathbb{R}^3 \times 0$



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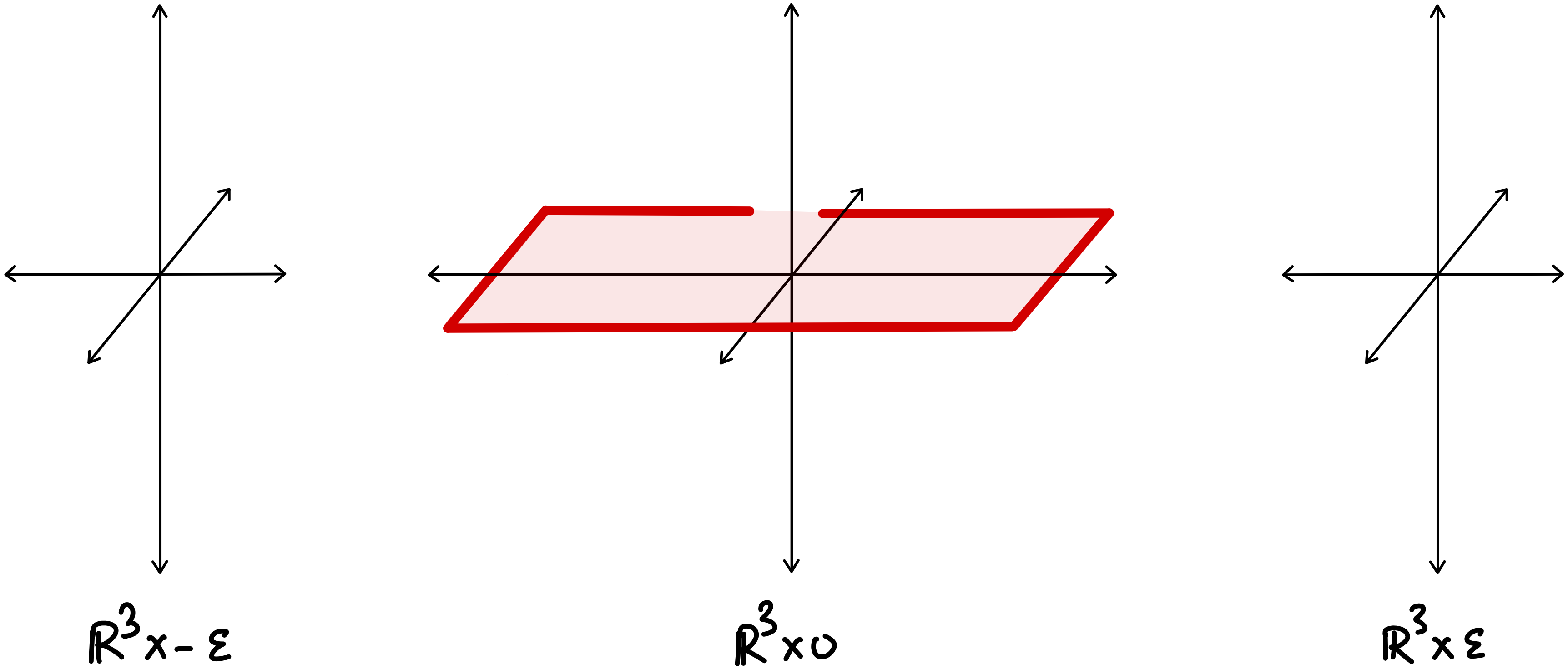
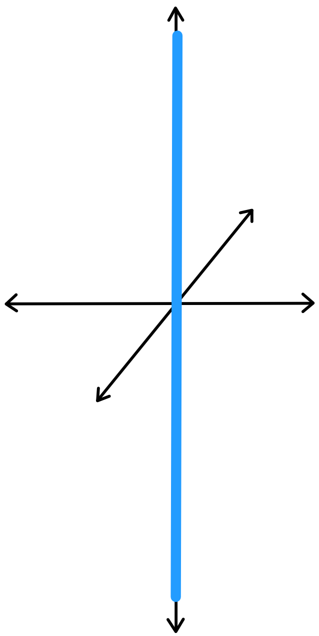
Visualising locally flat embeddings/generic immersions



Visualising locally flat embeddings/generic immersions



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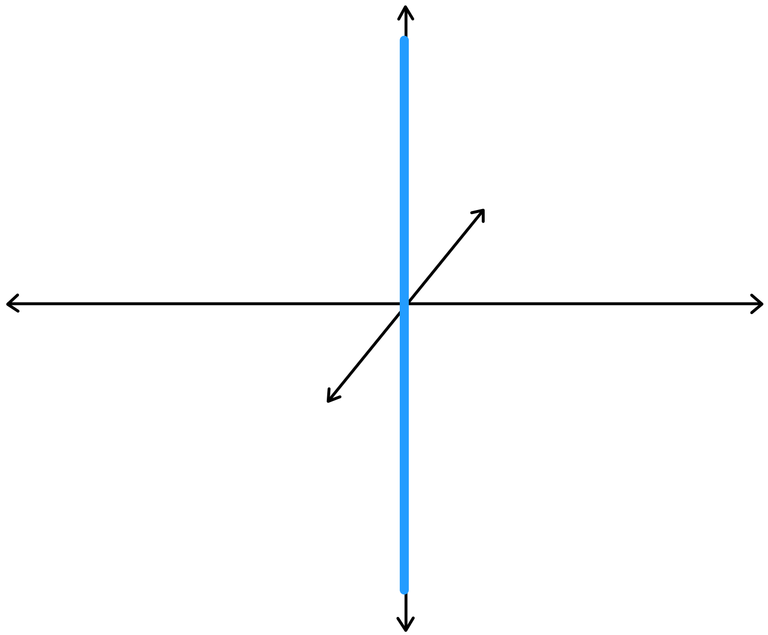
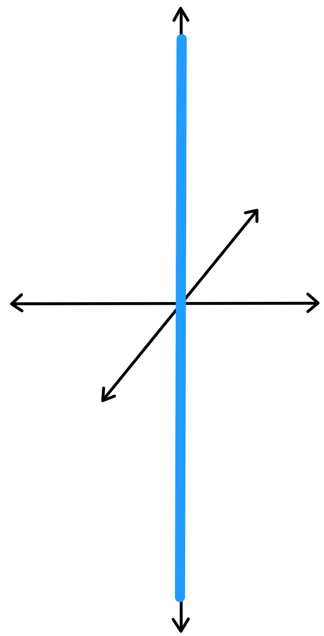
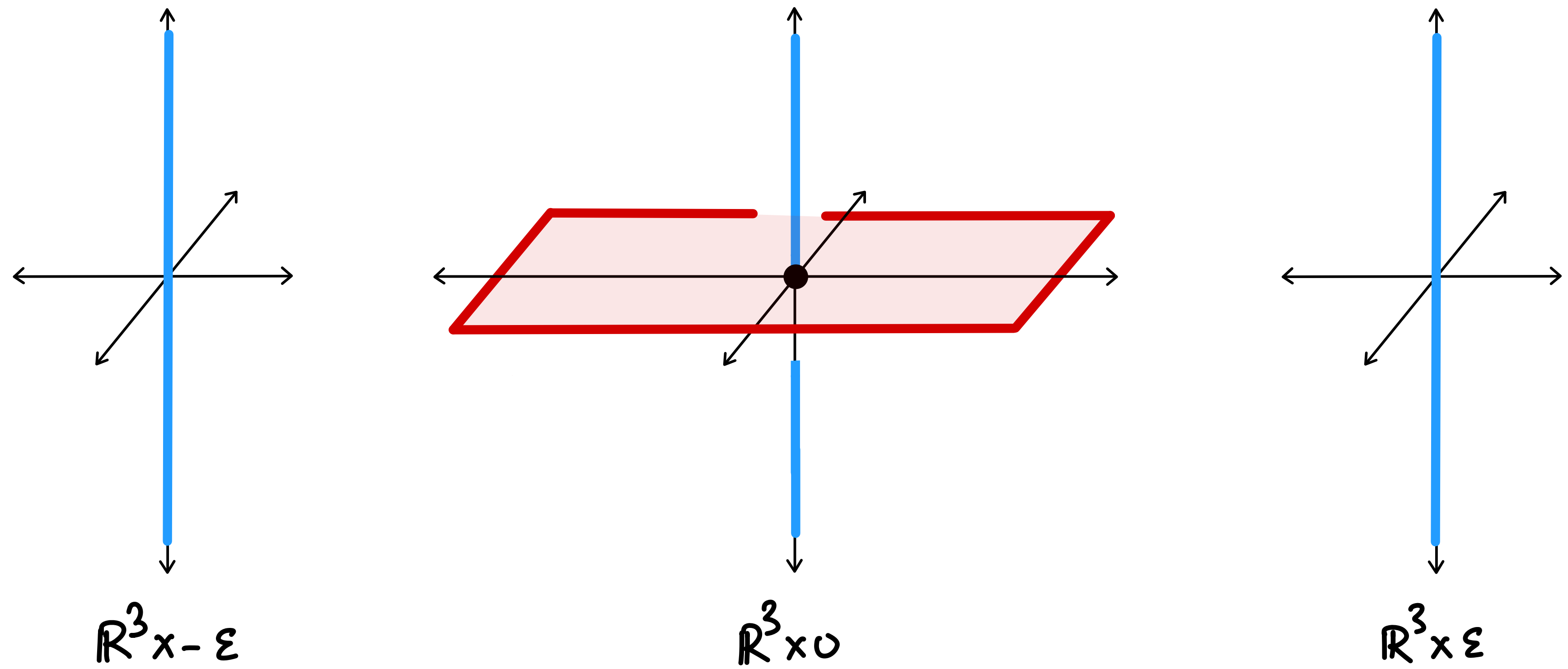


$\mathbb{R}^3 \times 0$

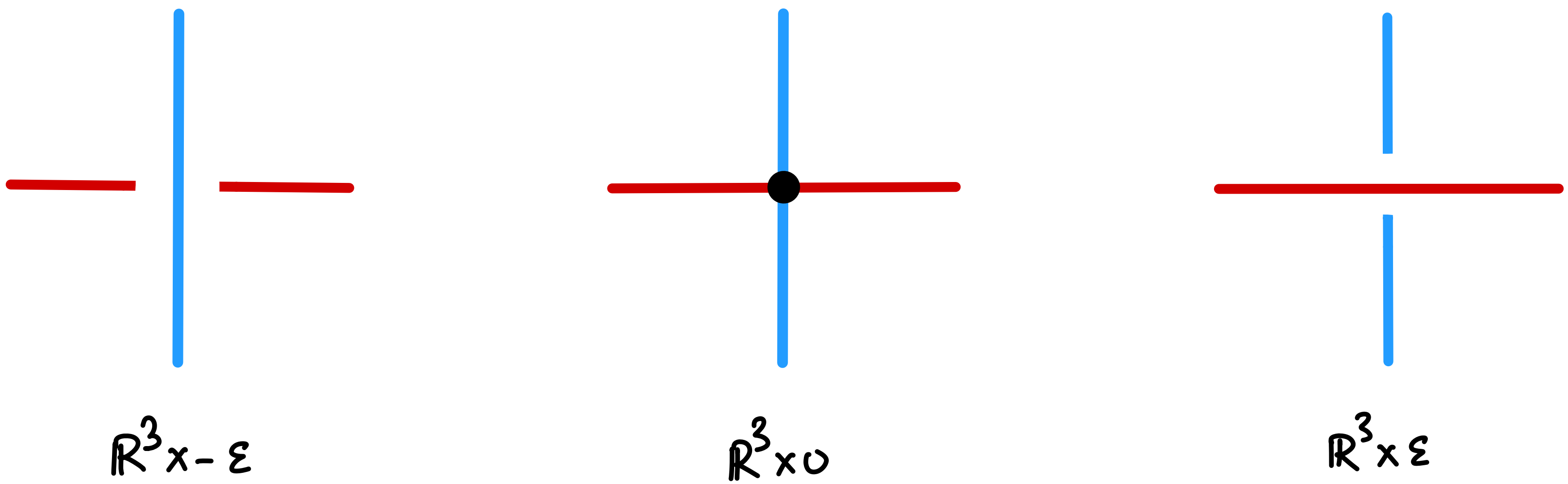


$\mathbb{R}^3 \times \varepsilon$

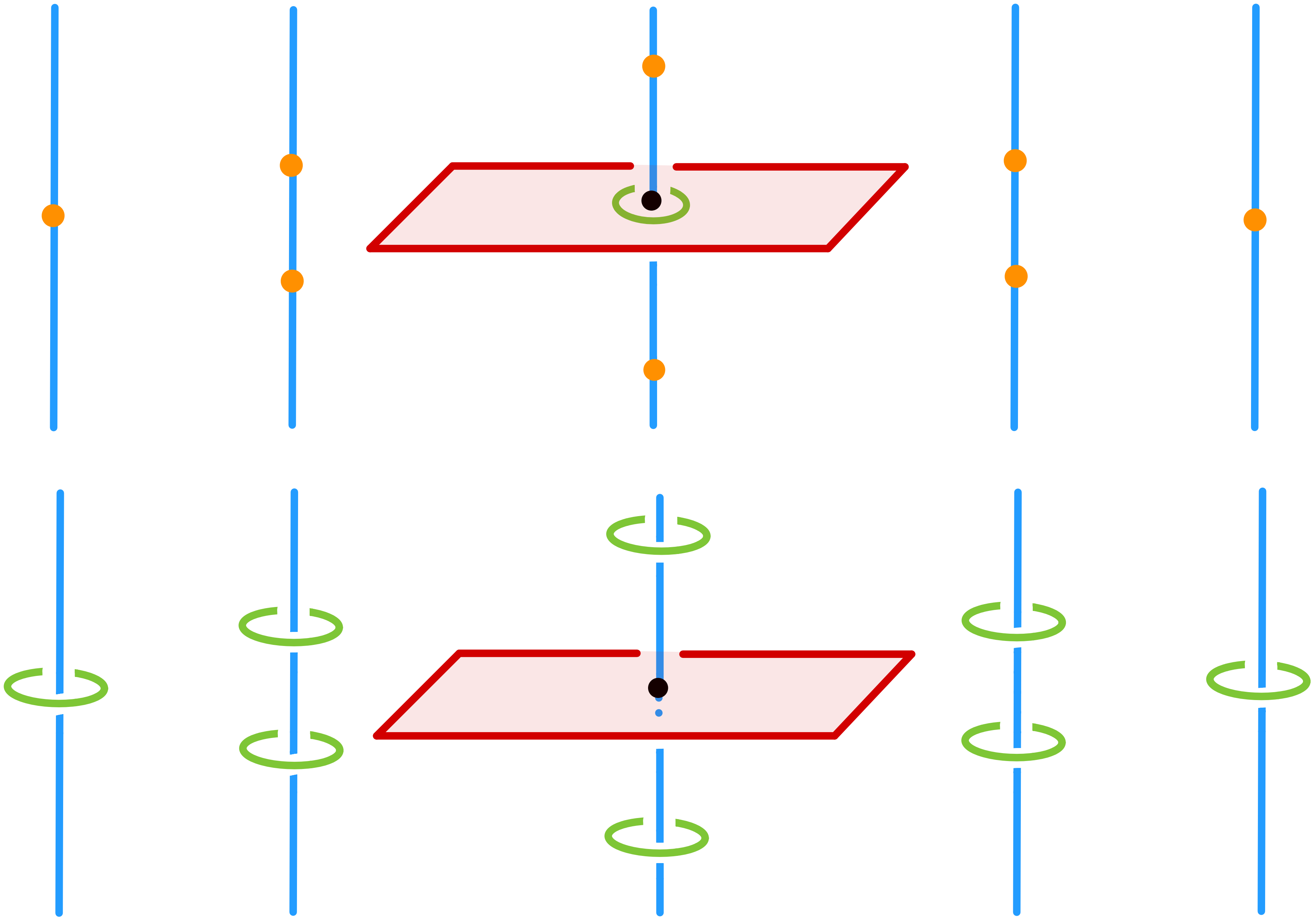
Visualising locally flat embeddings/generic immersions



Visualising locally flat embeddings/generic immersions



Visualising the Clifford torus $:= S^1_{xy} \times S^1_{zw}$



Fundamental tool: disc embedding theorem

Theorem [Freedman]: Let M be a simply connected 4-manifold.

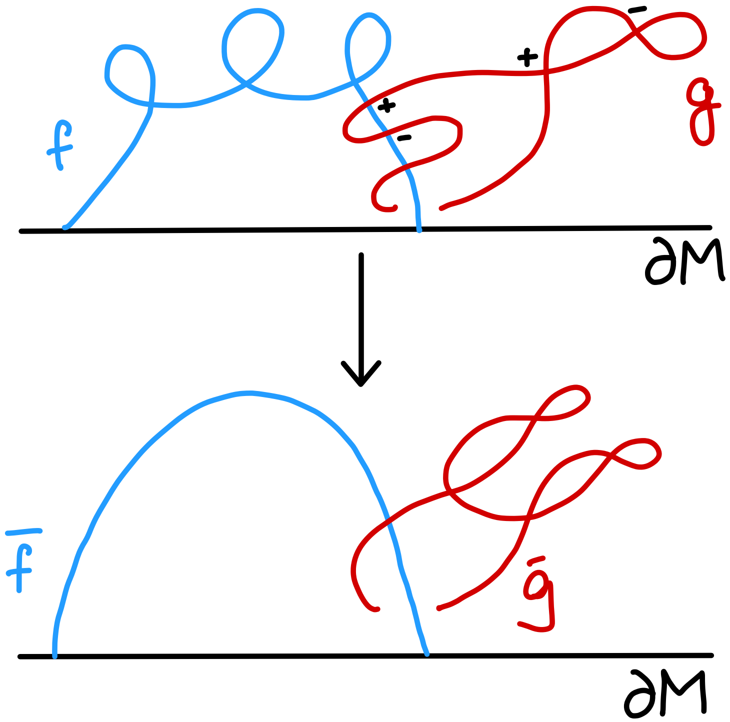
Let $f\colon D^2 \looparrowright M$ and $g\colon S^2 \looparrowright M$ be generic immersions such that:

$$\begin{tikzcd} D^2 \arrow[r, looparrowright] & M \\ \partial D^2 \arrow[u, hook] \arrow[r, hook] & \partial M \arrow[u, hook] \end{tikzcd}$$

(i) g has trivial normal bundle;

(ii) g has trivial self-intersection, $g.g = 0$;

(iii) f and g are algebraically dual, $f.g = 1$.



Then there exists $\bar{f} \simeq f$ and $\bar{g} \simeq g$ such that

(i) $\bar{f}$ is a locally flat embedding, and

(ii) $\bar{f}$ and $\bar{g}$ are geometrically dual, $\bar{f} \cap \bar{g}$ is a single, transverse double point.

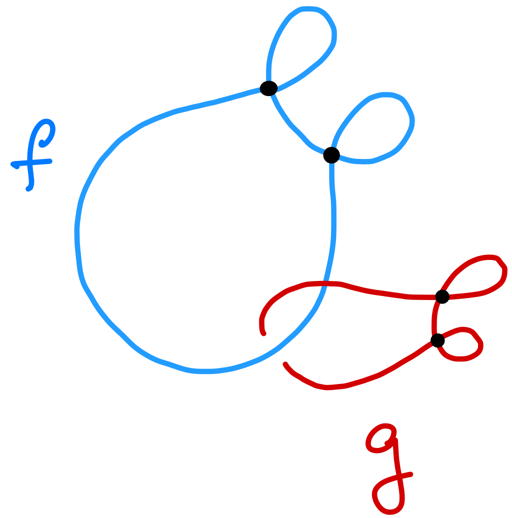
N.b. There was a typo on this slide during the actual talk!

Proof of the theorem

Theorem [Lee-Wilczyński]: Let M be a closed, simply connected 4-manifold. Then every primitive $\alpha \in H_2(M;\mathbb{Z})$ is represented by a locally flat embedded torus.

Proof:

Step 1: Represent α by a generic immersion $f\colon S^2 \looparrowright M$ with a geometrically dual sphere $g\colon S^2 \looparrowright M$.



$$\pi_1(M) = 1 \implies \pi_2(M) \cong H_2(M;\mathbb{Z})$$
$$\implies \text{there exists } f\colon S^2 \looparrowright M \text{ with } [f] = \alpha$$

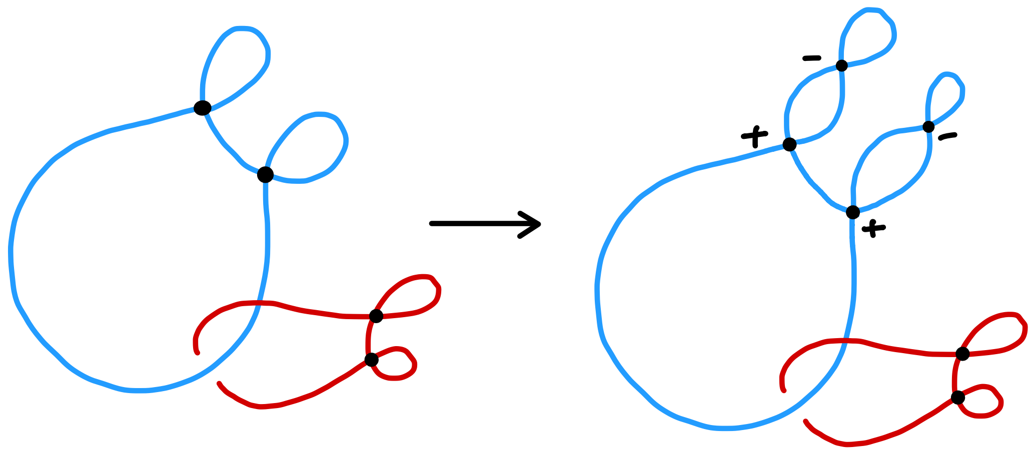
By Poincaré duality, there exists $\beta \in H_2(M;\mathbb{Z})$ with $\alpha \cdot \beta = 1$

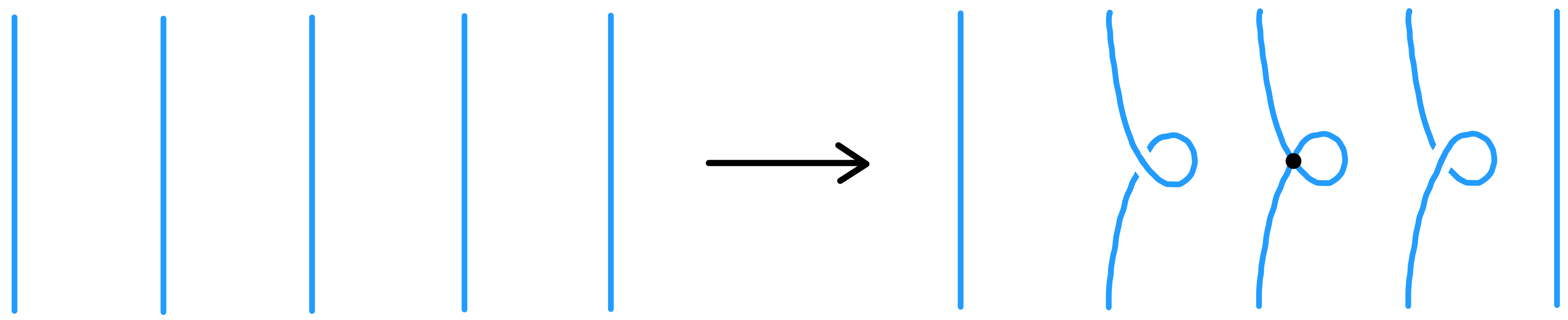
As before, there exists $g\colon S^2 \looparrowright M$ with $[g] = \beta$

Geometric Casson lemma: up to homotopy, can assume f and g are geometrically dual.

Proof of the theorem

Step 2: Arrange that the signed count of self-intersections of f is zero

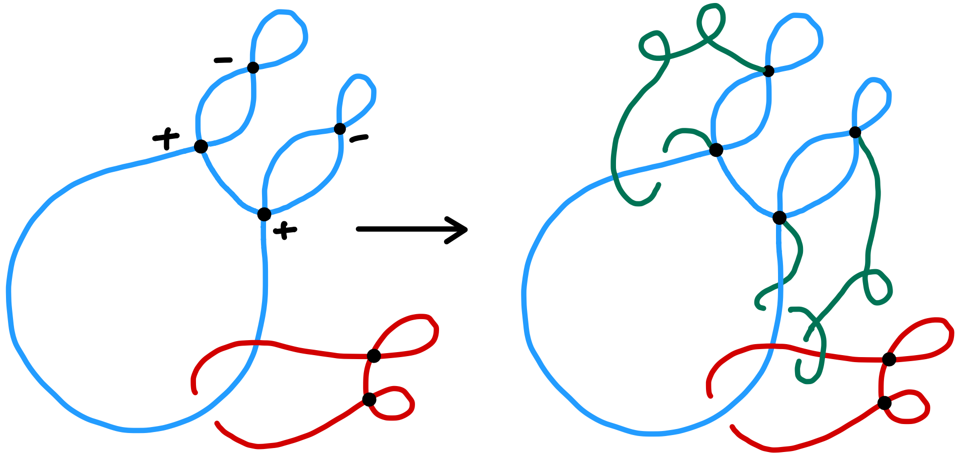


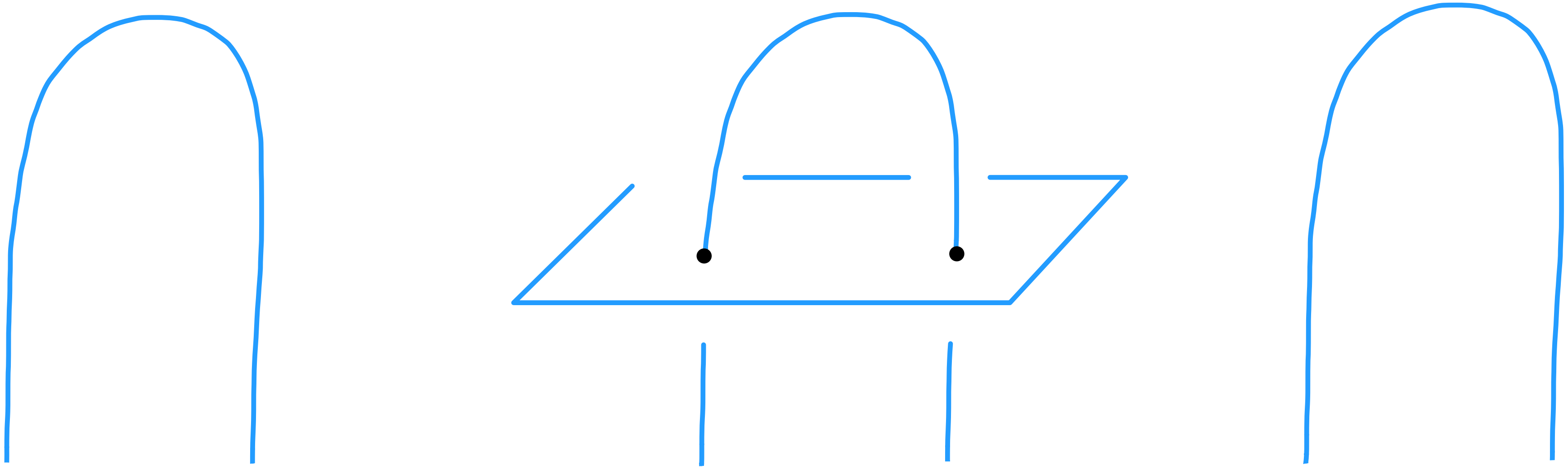


Interior twisting

Proof of the theorem

Step 3: Pair up self-intersections of f with generically immersed Whitney discs $\{W_i\}$





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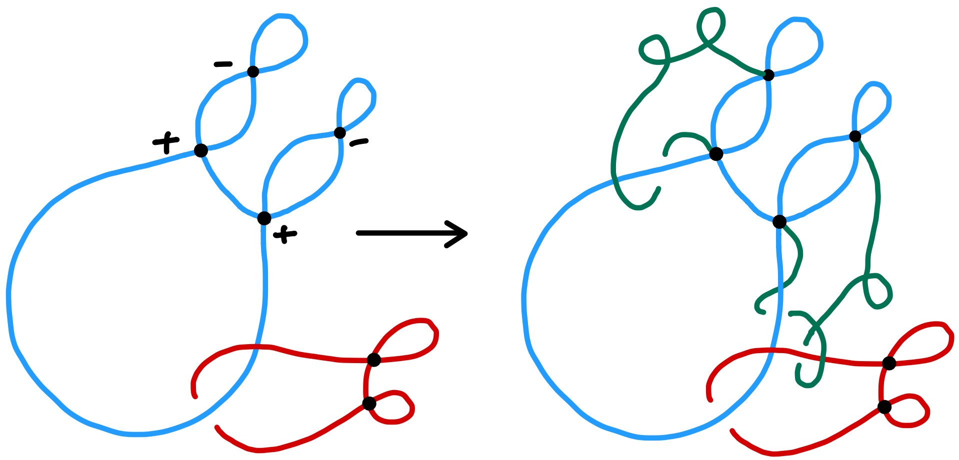

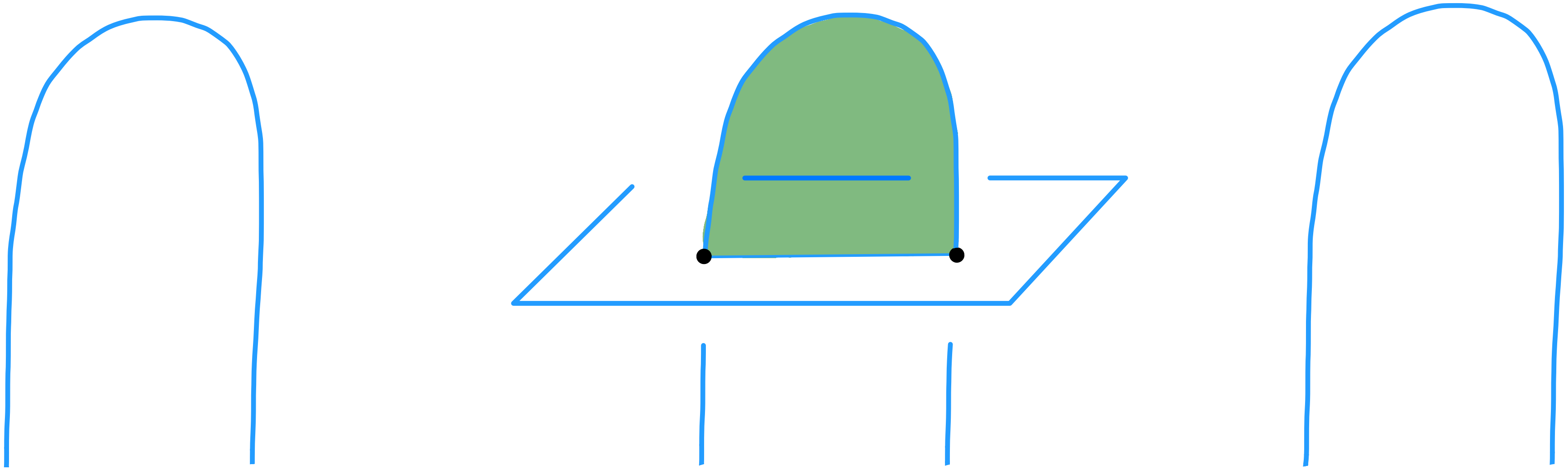

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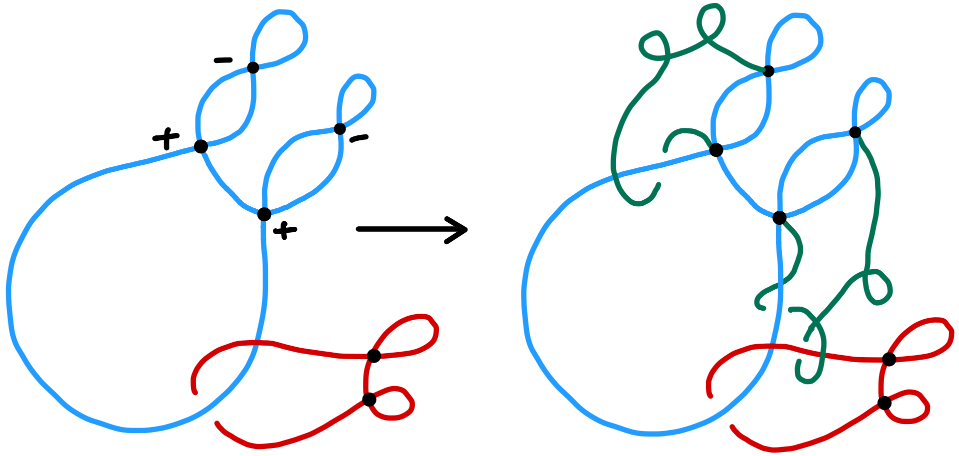
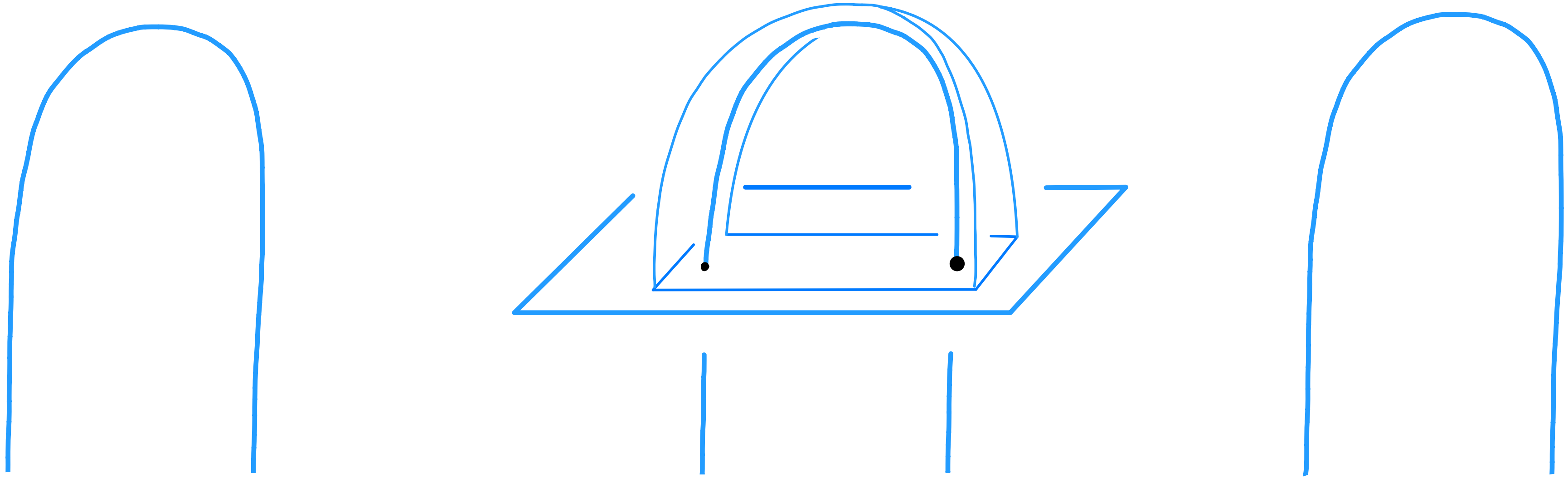
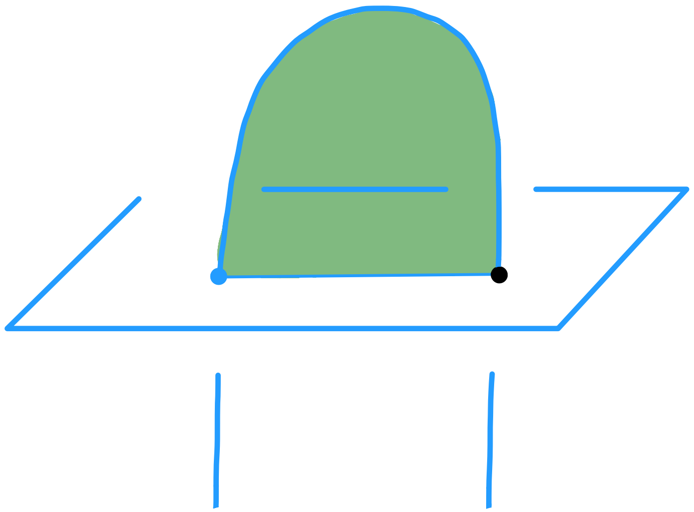
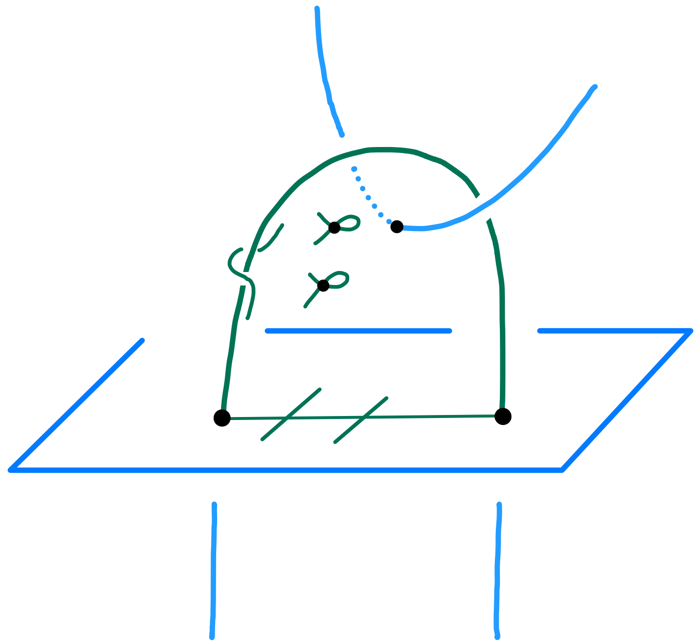




Proof of the theorem



Dream



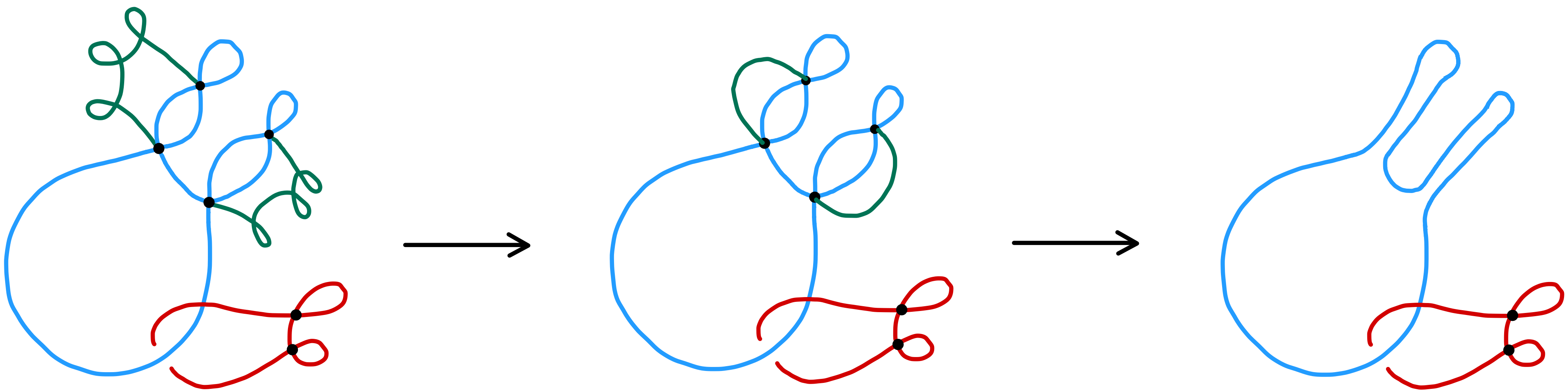
Reality

Proof of the theorem

Problem	Solution	Cost
$\mathring{W}_i \pitchfork \mathring{W}_j$	Disc embedding theorem	None if all else solved
$tw(\partial W_i)$	Interior twisting	$tw(\partial W_i) \to tw(\partial W_i) \pm 2$ $\lvert \mathring{W}_i \pitchfork \mathring{W}_i \rvert \to \lvert \mathring{W}_i \pitchfork \mathring{W}_i \rvert + 1$
	Boundary twisting	$tw(\partial W_i) \to tw(\partial W_i) \pm 1$ $\lvert \mathring{W}_i \pitchfork f \rvert \to \lvert \mathring{W}_i \pitchfork f \rvert + 1$
$\partial W_i \pitchfork \partial W_j$	Boundary pushoff	$\lvert \partial W_i \pitchfork \partial W_j \rvert \to \lvert \partial W_i \pitchfork \partial W_j \rvert - 1$ $\lvert \mathring{W}_i \pitchfork f \rvert \to \lvert \mathring{W}_i \pitchfork f \rvert + 1$
$\mathring{W}_i \pitchfork f$	Tubing into g	$\lvert \mathring{W}_i \pitchfork f \rvert \to \lvert \mathring{W}_i \pitchfork f \rvert - 1$ $tw(\partial W_i) \to tw(\partial W_i) + e(\nu g)$ $\lvert \mathring{W}_i \pitchfork \mathring{W}_j \rvert$ uncontrolled
	Transfer move	$\lvert \mathring{W}_i \pitchfork f \rvert \to \lvert \mathring{W}_i \pitchfork f \rvert + 1$ $\lvert \mathring{W}_j \pitchfork f \rvert \to \lvert \mathring{W}_j \pitchfork f \rvert + 1$

Proof of the theorem

Claim: If all self-intersections of f are paired by untwisted, generically immersed Whitney discs with embedded, disjoint boundaries and interiors disjoint from f, then can find locally flat embedded Whitney discs and do the Whitney move.



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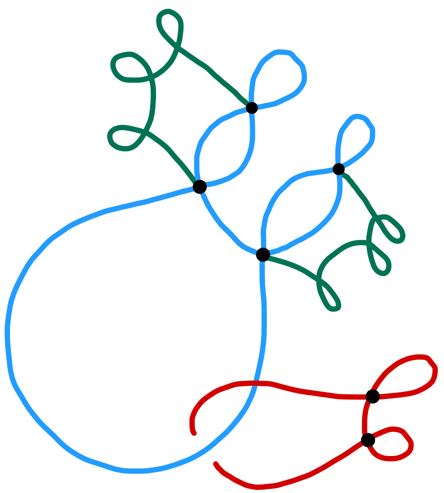

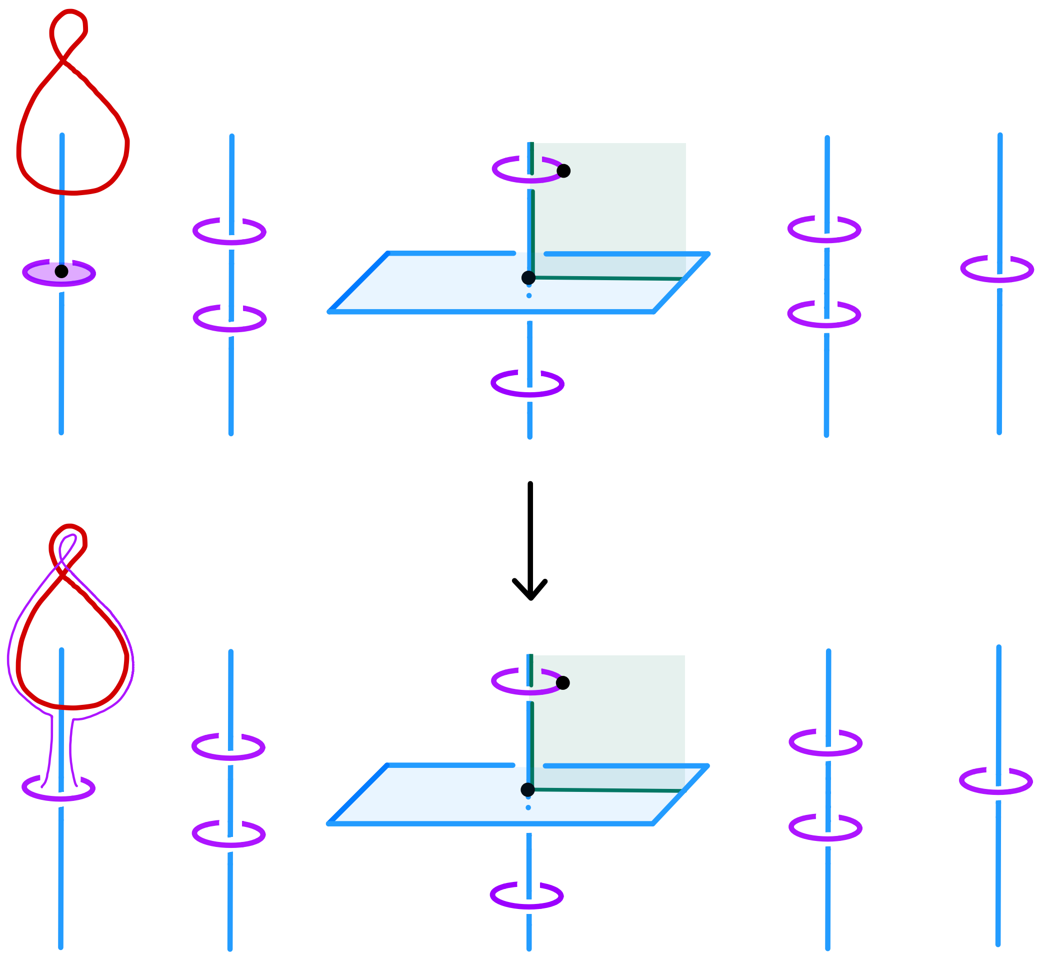

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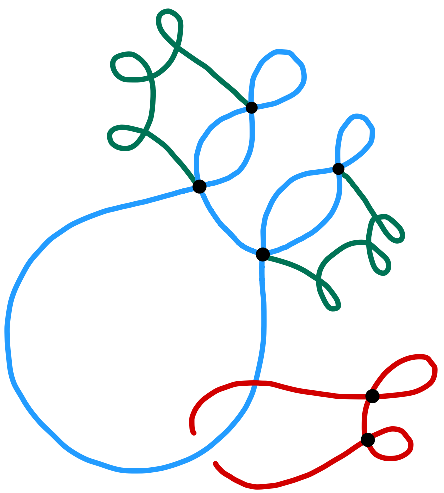

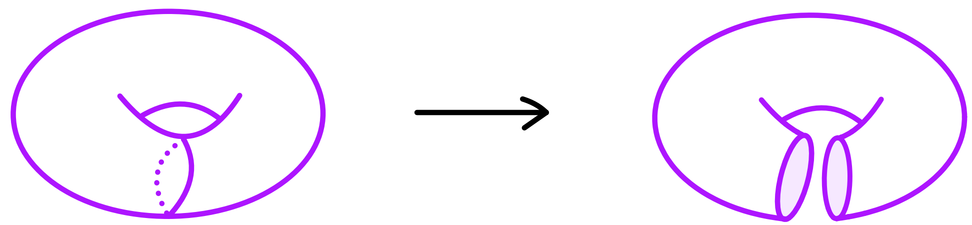

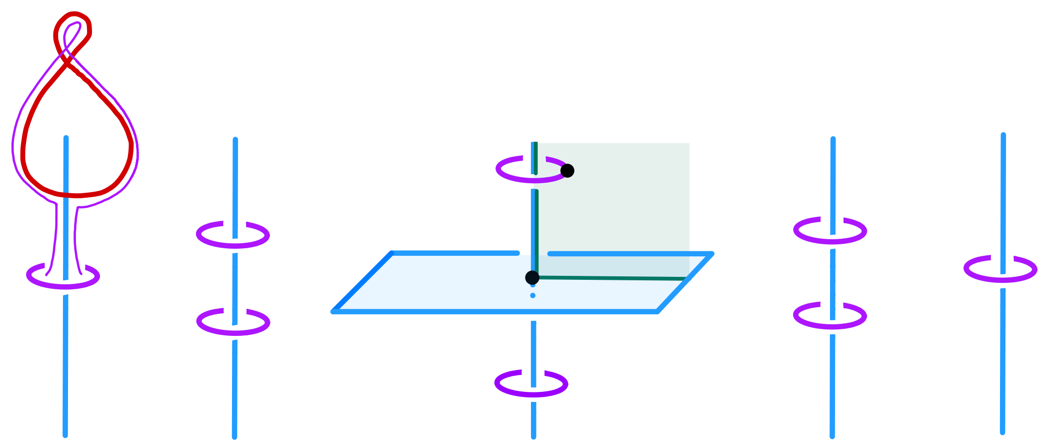

Proof of the theorem

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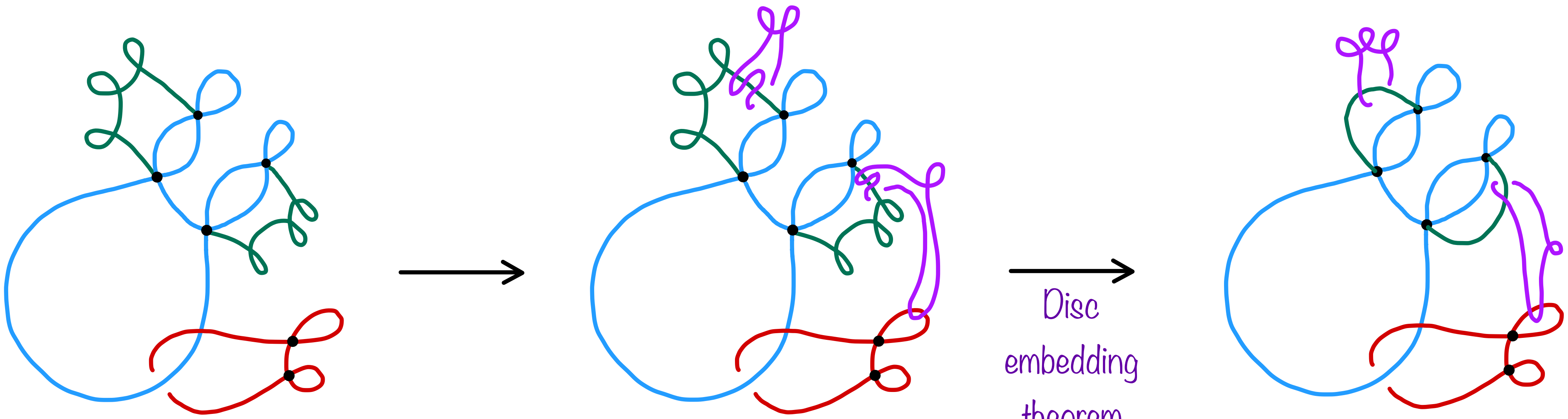

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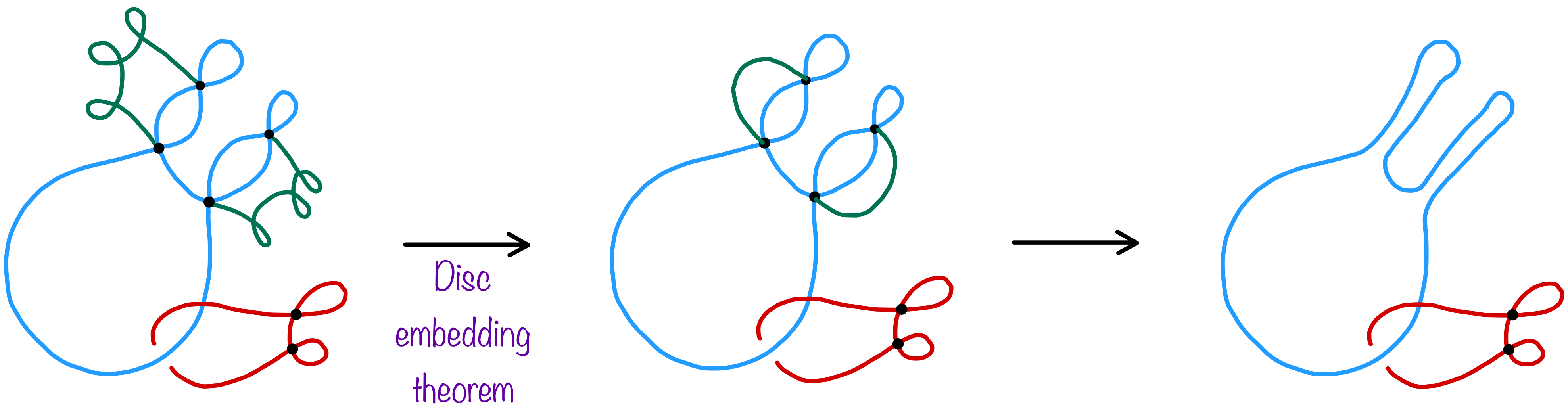


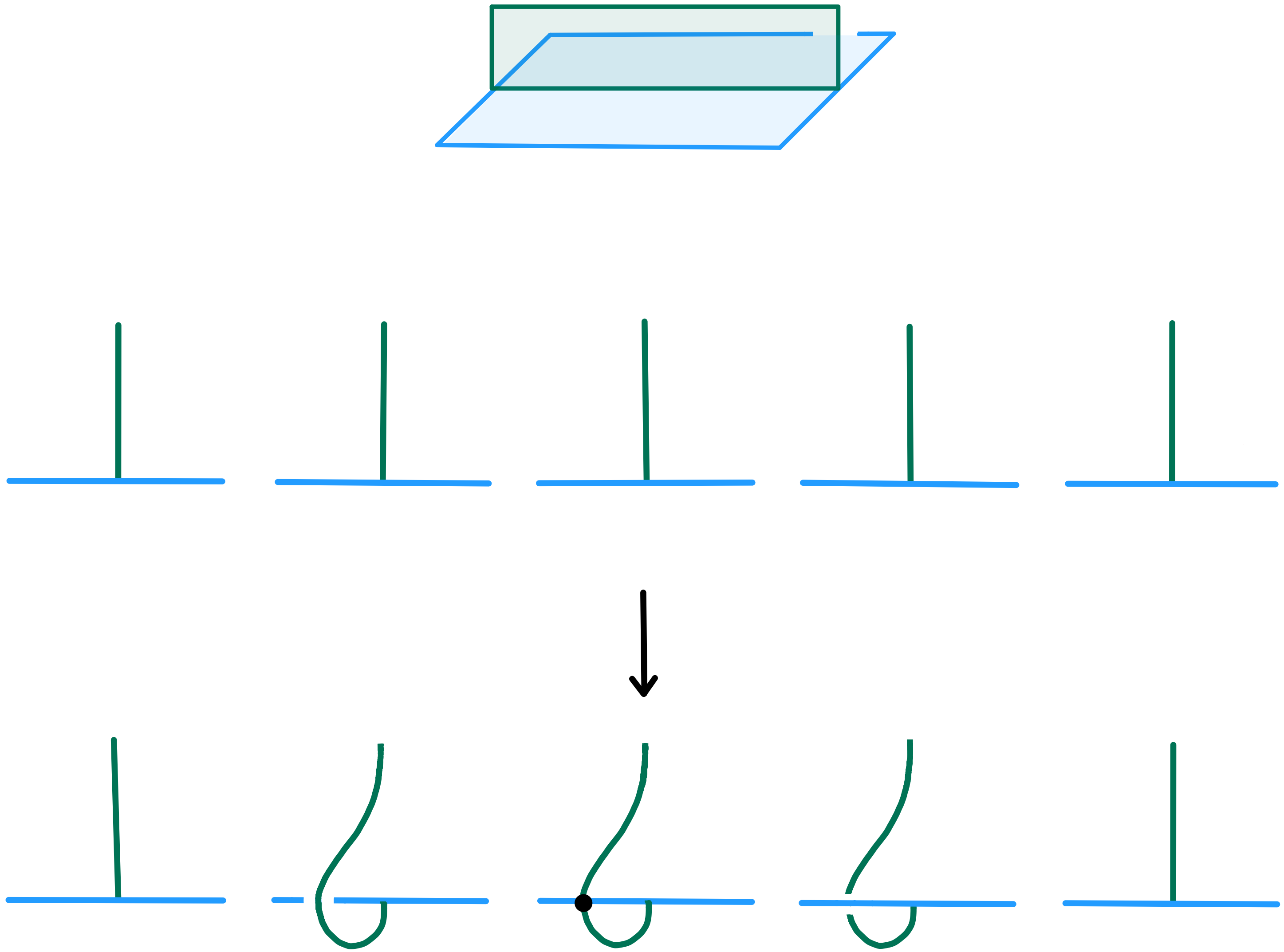
Proof of the theorem

Problem	Solution	Cost
$\mathring{W}_i \pitchfork \mathring{W}_j$	Disc embedding theorem	None if all else solved
	Interior twisting	
$tw(\partial W_i)$		
	Boundary twisting	
$\partial W_i \pitchfork \partial W_j$	Boundary pushoff	
	Tubing into g	
$\mathring{W}_i \pitchfork f$		
	Transfer move	

Proof of the theorem

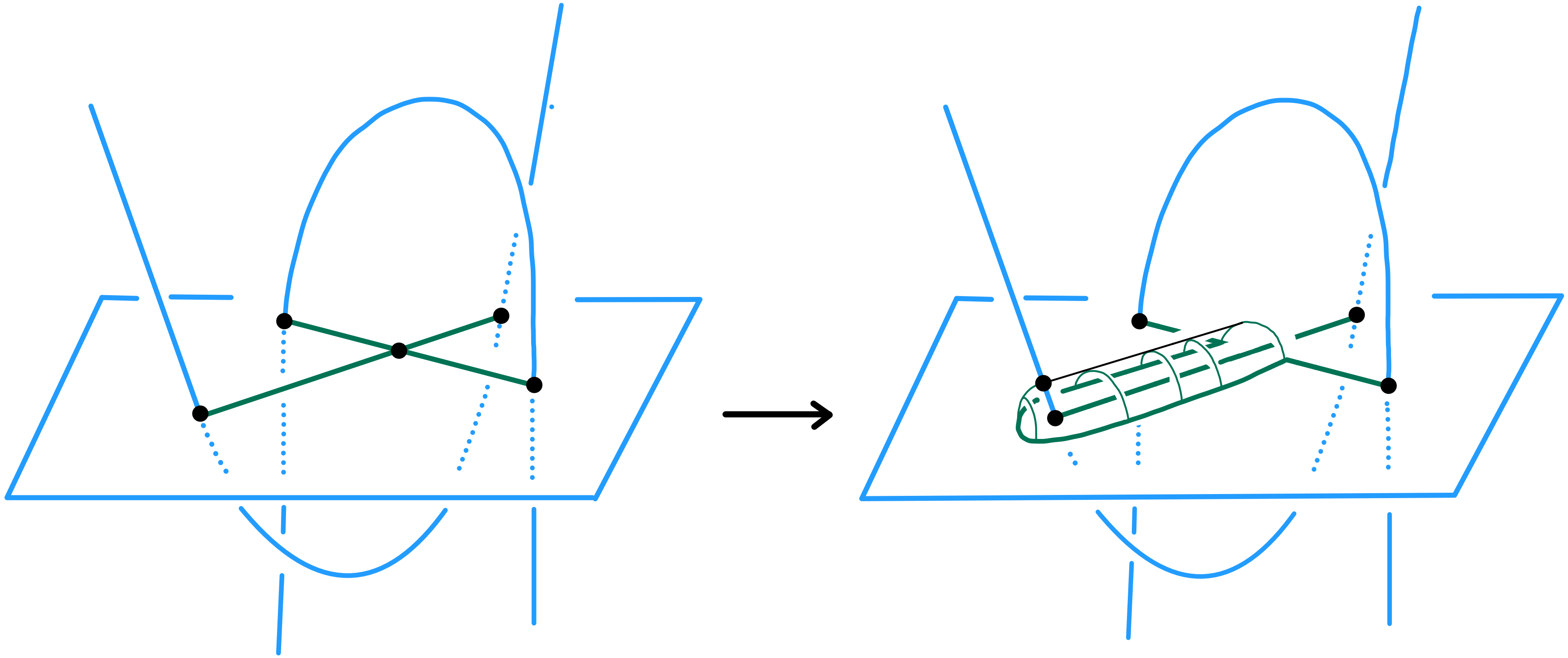
Problem	Solution	Cost
$\mathring{W}_i \pitchfork \mathring{W}_j$	Disc embedding theorem	None if all else solved
$tw(\partial W_i)$	Interior twisting	$tw(\partial W_i) \to tw(\partial W_i) \pm 2$ $\lvert\mathring{W}_i \pitchfork \mathring{W}_i\rvert \longrightarrow \lvert\mathring{W}_i \pitchfork \mathring{W}_i\rvert + 1$
	Boundary twisting	
$\partial W_i \pitchfork \partial W_j$	Boundary pushoff	
$\mathring{W}_i \pitchfork f$	Tubing into g	
	Transfer move	

Geometric manoeuvres



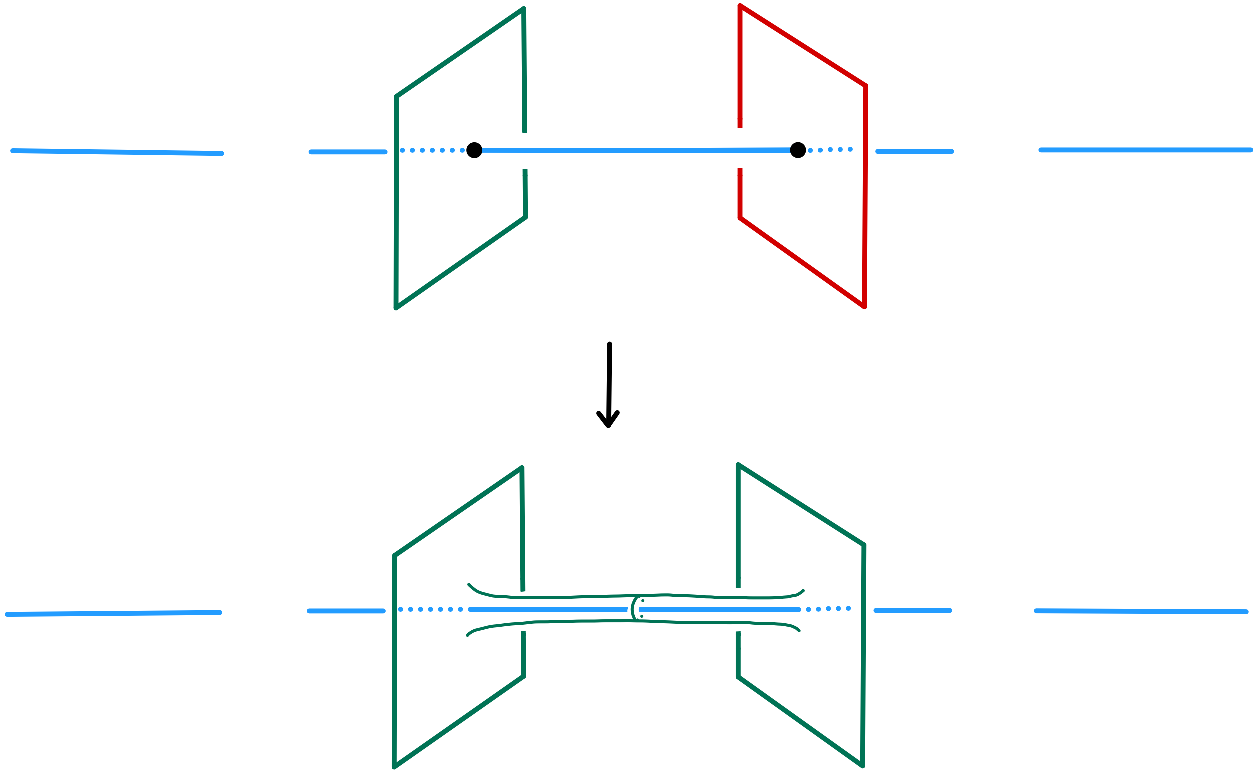
Boundary twisting

Geometric manoeuvres



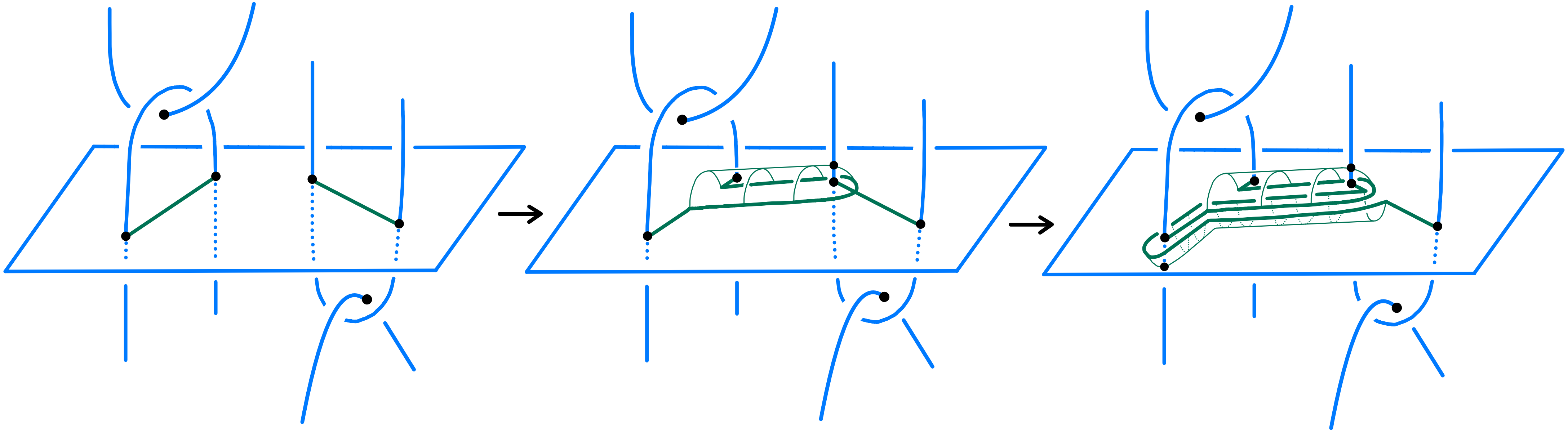
Boundary pushoff

Geometric manoeuvres



Tubing

Geometric manoeuvres



Transfer move

Proof of the theorem

Problem	Solution	Cost
$\mathring{W}_i \pitchfork \mathring{W}_j$	Disc embedding theorem	None if all else solved
$tw(\partial W_i)$	Interior twisting	$tw(\partial W_i) \to tw(\partial W_i) \pm 2$ $\lvert \mathring{W}_i \pitchfork \mathring{W}_i \rvert \to \lvert \mathring{W}_i \pitchfork \mathring{W}_i \rvert + 1$
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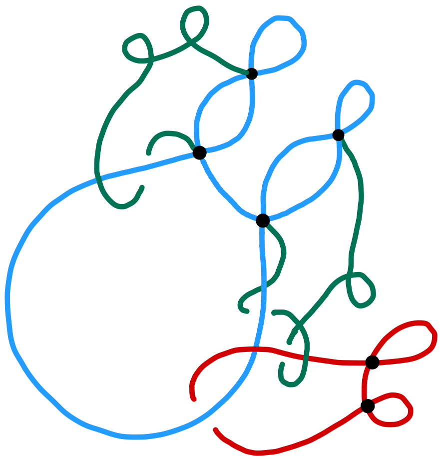
$\partial W_i \pitchfork \partial W_j$	Boundary pushoff	$\lvert \partial W_i \pitchfork \partial W_j \rvert \to \lvert \partial W_i \pitchfork \partial W_j \rvert - 1$ $\lvert \mathring{W}_i \pitchfork f \rvert \to \lvert \mathring{W}_i \pitchfork f \rvert + 1$
$\mathring{W}_i \pitchfork f$	Tubing into g	$\lvert \mathring{W}_i \pitchfork f \rvert \to \lvert \mathring{W}_i \pitchfork f \rvert - 1$ $tw(\partial W_i) \to tw(\partial W_i) + e(\nu g)$ $\lvert \mathring{W}_i \pitchfork \mathring{W}_j \rvert$ uncontrolled
	Transfer move	$\lvert \mathring{W}_i \pitchfork f \rvert \to \lvert \mathring{W}_i \pitchfork f \rvert + 1$ $\lvert \mathring{W}_j \pitchfork f \rvert \to \lvert \mathring{W}_j \pitchfork f \rvert + 1$

Returning to proof of the theorem

Step 4: Use geometric manoeuvres to ensure $\{W_i\}$ are untwisted and have embedded, disjoint boundaries, and moreover $\{\mathring{W}_i \pitchfork f\}$ has at most one point.



$\xrightarrow{\text{Boundary pushoff}}$ $\{\partial W_i\}$ embedded, disjoint $\xrightarrow{\text{Interior twisting}}$ $tw(\partial W_i) = 0$ or 1 for each i

$\Big\downarrow$ Boundary twisting

$tw(\partial W_i) = 0$ for each i $\xrightarrow{\text{Transfer move}}$ $\mathring{W}_i \pitchfork f$ even for all but possibly one i, say $i = 0$

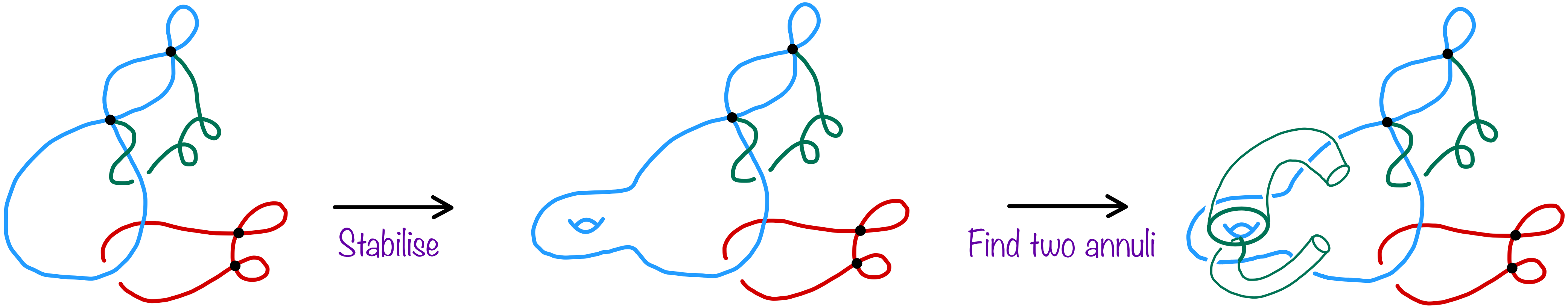
$\Big\downarrow$ Tube all W_i with $i \neq 0$ into g

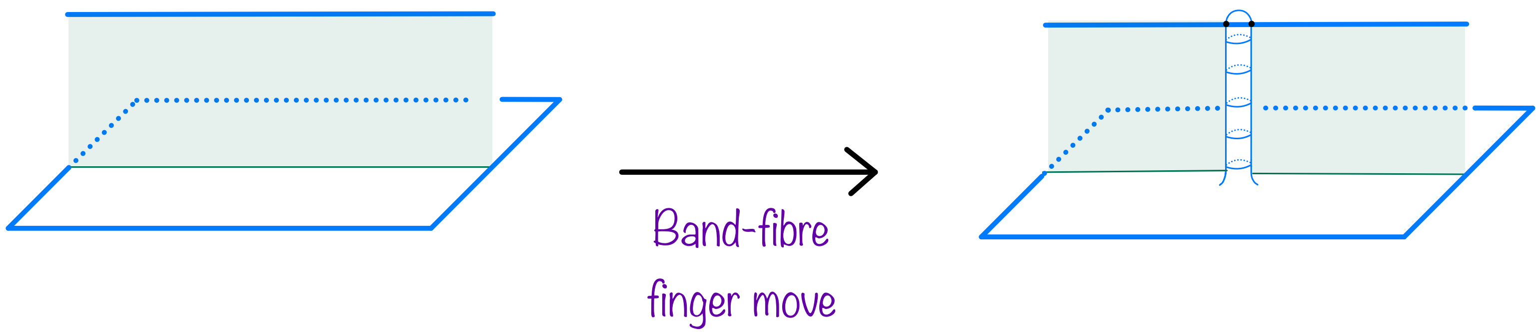
$tw(\partial W_i)$ even for all $i \neq 0$ $\xrightarrow{\text{Interior twisting}}$ $\{W_i\}_{i \neq 0}$ untwisted, embedded, disjoint boundaries, interiors disjoint from f

Returning to proof of the theorem

Step 5: If $\{W_i\}$ untwisted, with embedded disjoint boundaries, and interior disjoint from f, for all i, proceed to next step.

If not: stabilise to add genus and do the band-fibre finger move twice.





Returning to proof of the theorem

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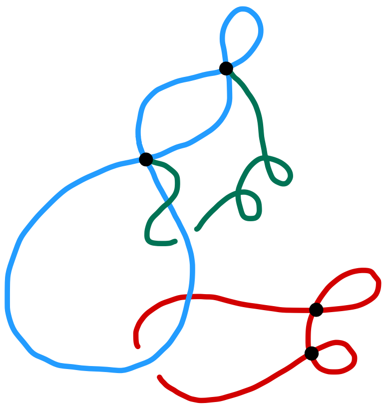
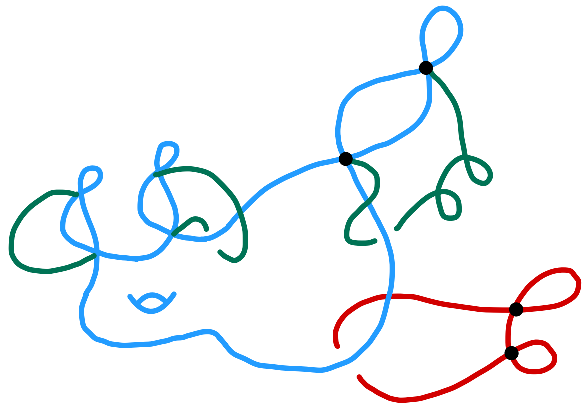


Two new Whitney discs V_1 and V_2, with $\partial V_1 \pitchfork \partial V_2$ a single point

Boundary pushoff

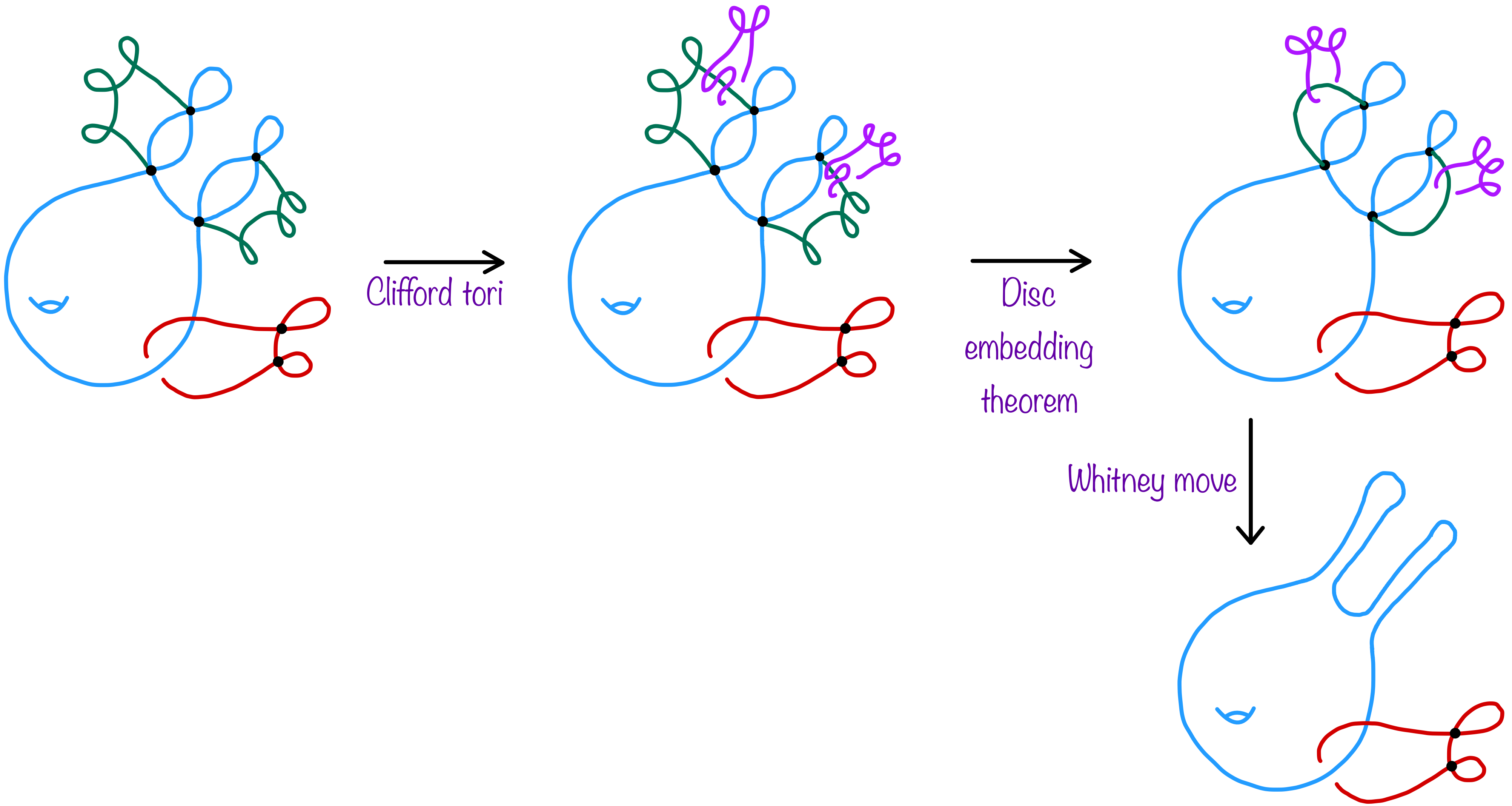


Transfer move, tube into g, interior twist

$\{W_i\}$ untwisted, with embedded disjoint boundaries, and interior disjoint from f, for all i.

Returning to proof of the theorem

Step 6: Use Clifford tori to get algebraically dual spheres for $\{W_i\}$, apply the disc embedding theorem to find locally flat embedded Whitney discs, and do the Whitney move.



$\square$

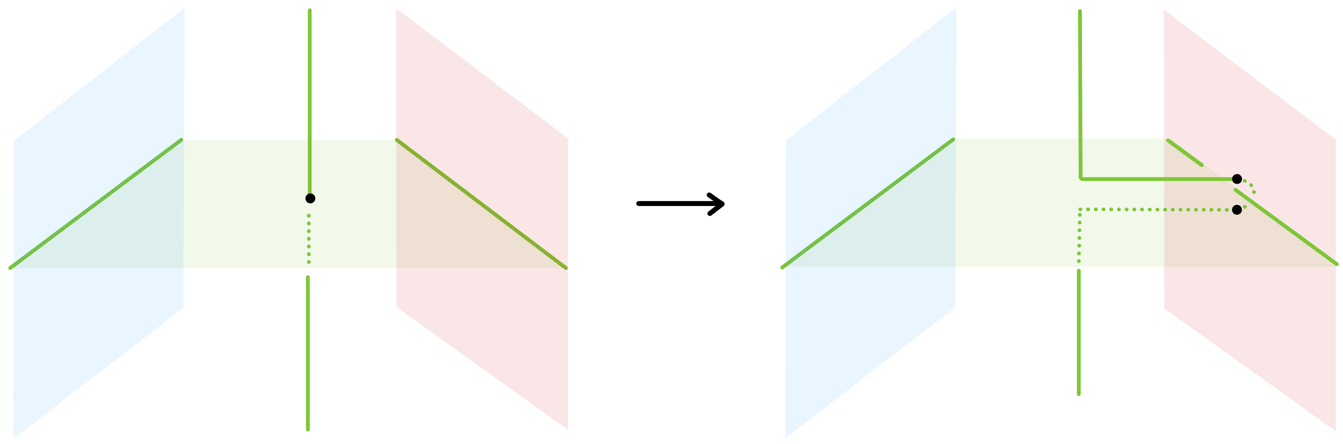
Locally flat surfaces in 4-manifolds

Talk 1:

Theorem [Lee-Wilczyński]: Let M be a closed, simply connected 4-manifold. Then every primitive $\alpha \in H_2(M;\mathbb{Z})$ is represented by a locally flat embedded torus.

Talk 2 (next week):

Theorem [Freedman-Quinn]: Let $K \subseteq S^3$ be a knot with Alexander polynomial one. Then K is topologically slice.



Geometric Casson lemma

Moscow-Beijing topology seminar
June 25, 2025

Constructing locally flat surfaces in 4-manifolds

Locally flat surfaces in 4-manifolds

Talk 1 (last week):

Theorem [Lee-Wilczyński]: Let M be a closed, simply connected 4-manifold.
Then every primitive $\alpha \in H_2(M;\mathbb{Z})$ is represented by a locally flat embedded torus.

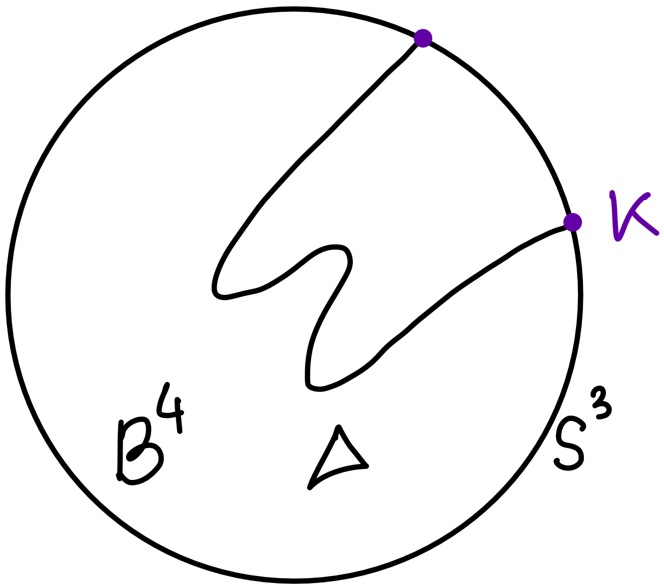
Talk 2 (today):

Theorem [Freedman-Quinn]: Let $K \subseteq S^3$ be a knot with Alexander polynomial one.
Then K is topologically slice.

Topologically slice knots

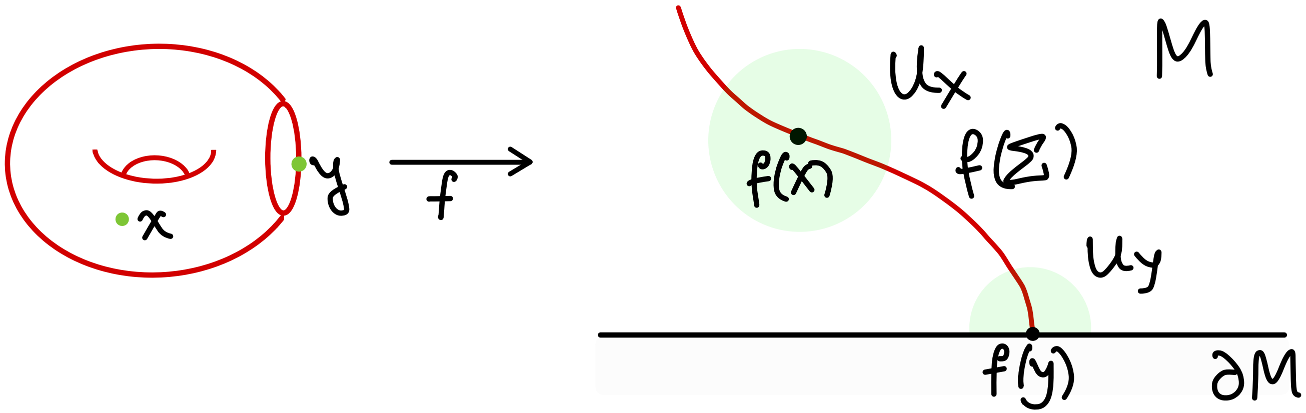
Definition [Fox-Milnor]: A knot $K \subseteq S^3$ is said to be topologically slice if it bounds a locally flat disc in B^4, i.e.

$$\begin{tikzcd} S^1 \arrow[r, hook, "K"] \arrow[d, hook, "\partial"'] & S^3 \arrow[d, hook, "\partial"] \\ D^2 \arrow[r, hook, "\Delta", "\text{locally flat}"'] & B^4 \end{tikzcd}$$



Locally flat surfaces in 4-manifolds

Definition: Let M be a 4-manifold and Σ a surface. An embedding $f: \Sigma \hookrightarrow M$ is said to be *locally flat* if $\forall\, x \in \Sigma$, there exists $U \ni f(x)$ with $(U, U \cap f(\Sigma)) \approx (\mathbb{R}^4, \mathbb{R}^2)$



Fundamental tools

Theorem [Freedman-Quinn]: Every continuous map $\Sigma \longrightarrow M$ is homotopic to a generic immersion.

Theorem [Quinn]: Every locally flat submanifold of M has a (linear) normal bundle.

Theorem [Quinn]: Let Σ_1, Σ_2 denote locally flat submanifolds of M. Then there is an ambient isotopy of M taking Σ_1 to Σ_1' such that Σ_1' and Σ_2 intersect transversely.

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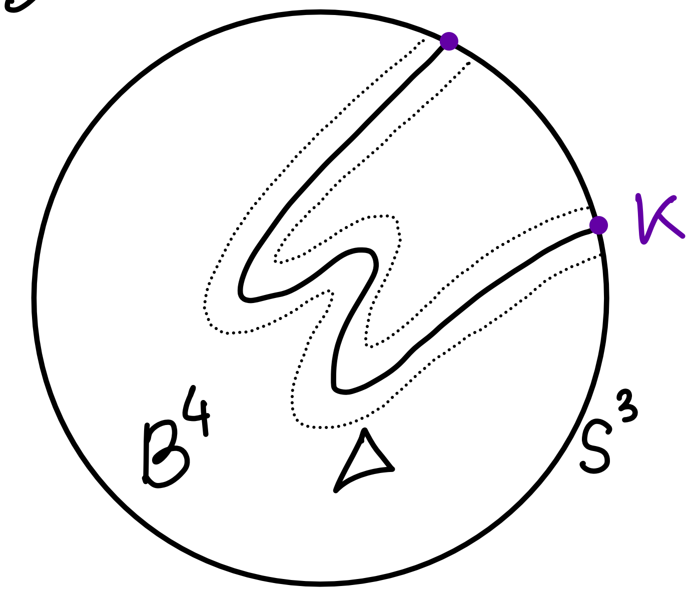

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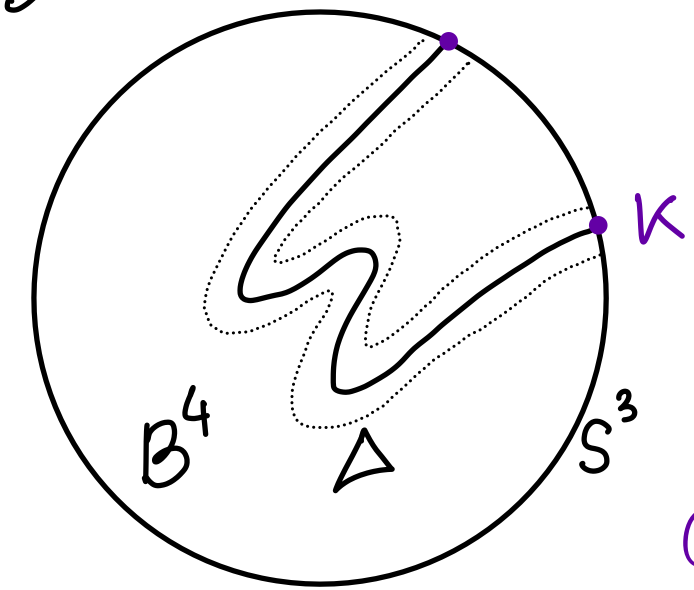


Check:

$\partial(B^4 \setminus \mathring{\nu}\Delta) = S^3_0(K)$

0-framed Dehn surgery on K

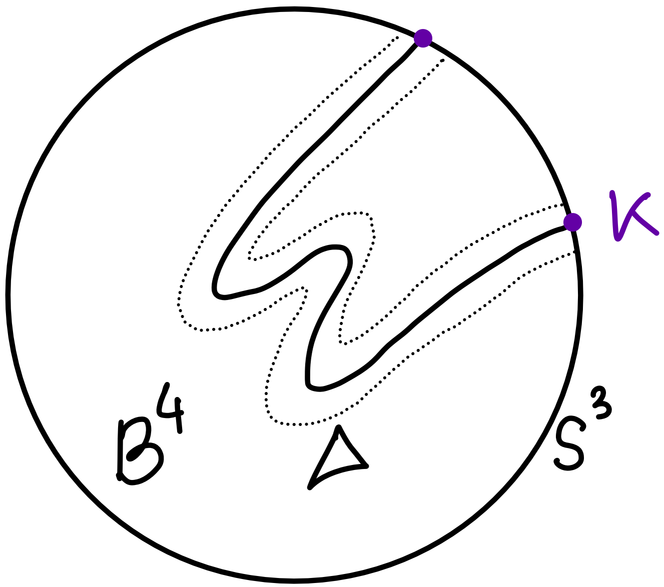
0-surgery characterisation of topological sliceness

Theorem: A knot $K \subseteq S^3$ is topologically slice if and only if $S^3_0(K) = \partial W$ where W is a compact, connected topological 4-manifold such that

(i) $\mathbb{Z} \cong H_1(S^3_0(K;\mathbb{Z}) \xrightarrow[\cong]{\iota_*} H_1(W;\mathbb{Z})$;

(ii) $\pi_1 W$ is normally generated by the meridian $\mu_K \subseteq S^3_0(K)$; and

(iii) $H_2(W;\mathbb{Z}) = 0$.



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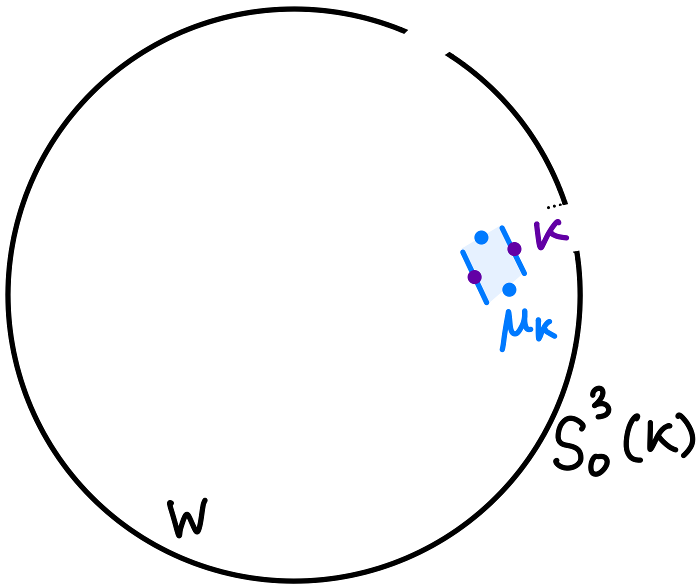


Proof of "if" direction:

- Given W glue on $D^2 \times D^2$ along μ_K
- Identify resulting manifold as B^4
- Identify $0 \times D^2$ as topological slice disc for K

Alexander polynomial one knots

Proposition/definition: A knot $K \subseteq S^3$ has Alexander polynomial one if and only if

$$H_1(S^3_0(K); \mathbb{Z}[\mathbb{Z}]) = 0$$

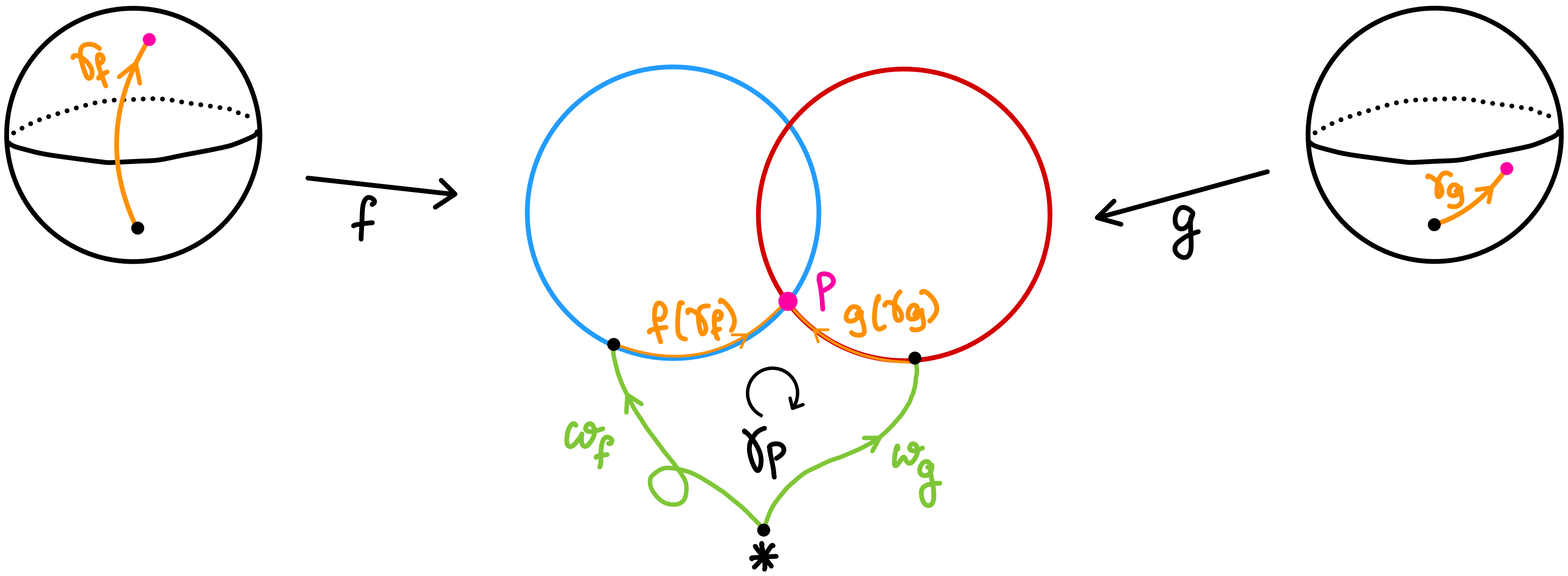
Note: $H_1(S^3_0(K); \mathbb{Z}) \cong \mathbb{Z}$ so $S^3_0(K)$ has an infinite cyclic cover $\widetilde{S^3_0(K)}$.

By definition, as an abelian group $H_1(S^3_0(K); \mathbb{Z}[\mathbb{Z}])$ is simply $H_1(\widetilde{S^3_0(K)}; \mathbb{Z})$.
Remembering the action of $\mathbb{Z}$ makes this a $\mathbb{Z}[\mathbb{Z}]$-module.

Equivariant intersection numbers

Definition: Let M be a closed, oriented, connected 4-manifold with a base point $* \in M$.

Define $\lambda_M : \pi_2(M) \times \pi_2(M) \longrightarrow \mathbb{Z}[\pi_1 M]$ as follows.



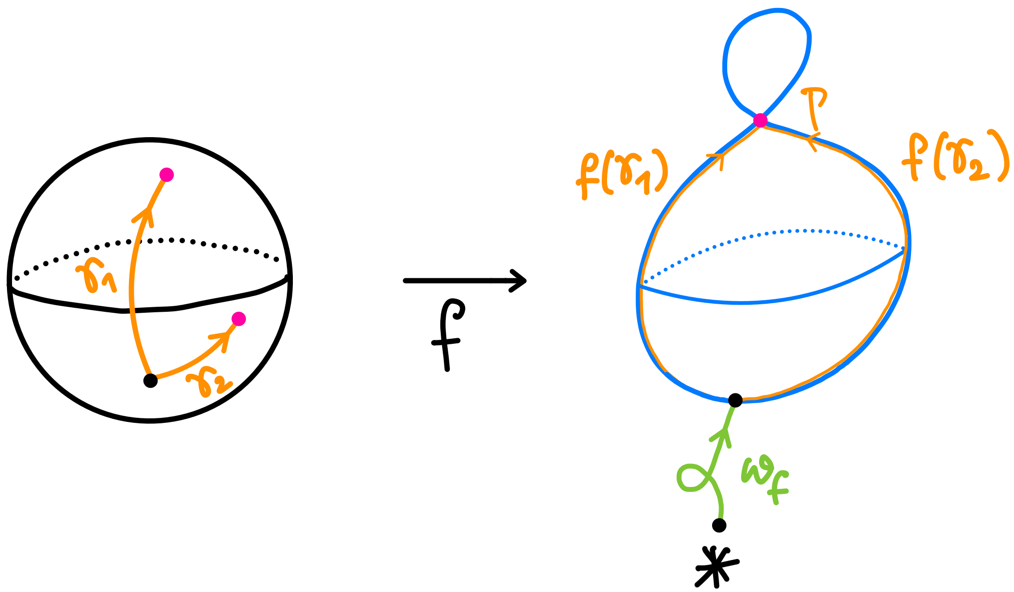
$$\lambda_M(f,g) = \sum_{p \in f \pitchfork g} \varepsilon_p \cdot \gamma_p \in \mathbb{Z}[\pi_1 M]$$

$\{\pm 1\}$ sign of p

Equivariant self-intersection numbers

Definition: Let M be a closed, oriented, connected 4-manifold with a base point $* \in M$.

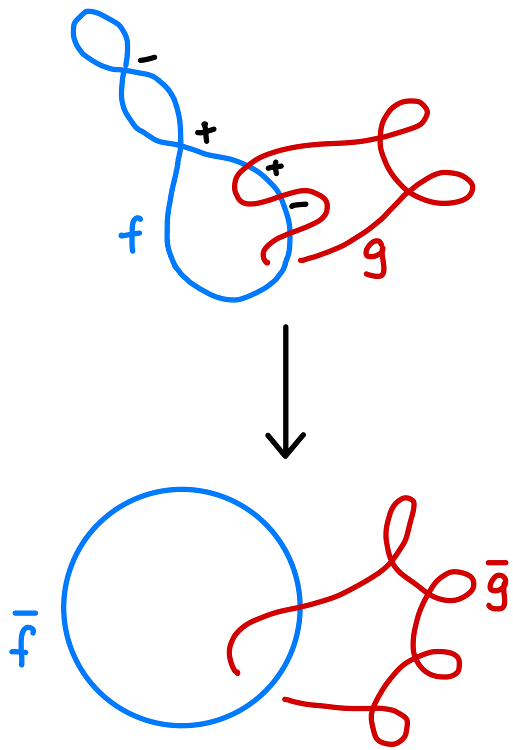
Define $\mu_M : \pi_2(M) \longrightarrow \mathbb{Z}[\pi_1 M]/\sim$ as follows.



$$\mu_M(f) = \sum_{p \in f \pitchfork f} \varepsilon_p \cdot \gamma_p \in \mathbb{Z}[\pi_1 M]/\sim$$

Fundamental tool: sphere embedding theorem

Theorem [Freedman-Quinn]: Let M be a connected 4-manifold.
Suppose that $\pi_1(M)$ is "good", e.g. finite, abelian, solvable, ...
Let $f\colon S^2 \looparrowright M$ and $g\colon S^2 \looparrowright M$ be generic immersions such that:



(i) g has trivial normal bundle;
(ii) $\mu_M(f) = 0$; and
(iii) f and g are algebraically dual, $\lambda_M(f,g) = 1$.

Then there exist $\bar{f} \simeq f$ and $\bar{g} \simeq g$ such that
(i) $\bar{f}$ is a locally flat embedding, and
(ii) $\bar{f}$ and $\bar{g}$ are geometrically dual, $\bar{f} \cap \bar{g}$ is a single, transverse double point.

When $\pi_1(M) \neq 1$, outcome (ii) is due to Powell-R.-Teichner.

Proof of the theorem

Theorem [Freedman-Quinn]: Let $K \subseteq S^3$ be a knot with Alexander polynomial one. Then K is topologically slice.

Proof sketch:

Since $H_1(S^3_0(K); \mathbb{Z}) \cong \mathbb{Z}\langle \mu_K \rangle$, there is a map $f\colon S^3_0(K) \longrightarrow S^1$, such that the induced map on fundamental groups sends $[\mu_K] \longmapsto 1$.

Goal: Build a compact, connected 4-manifold W such that

$$\begin{tikzcd} S^3_0(K) \arrow[r, "f"] \arrow[d, hook, "\partial"'] & S^1 \\ W \arrow[ur, "\simeq"'] & \end{tikzcd}$$

Proof of the theorem

Step 1: Build compact, connected, spin 4-manifold V such that
$$\begin{tikzcd} S^3_0(K) \arrow[r, "f"] \arrow[d, hook, "\partial"'] & S^1 \\ V \arrow[ur, "F"'] & \end{tikzcd}$$

Input: $\Omega_3^{spin}(S^1) \cong \mathbb{Z}/2$
$$\begin{array}{c} \Omega_3^{spin}(S^1) \\ \rotatebox{90}{$\in$} \\ (S^3_0(K), \mathfrak{s}) \mapsto \mathrm{Arf}(K) = 0 \end{array}$$

Recall: Elements of $\Omega_3^{spin}(S^1)$ are $(Y, \mathfrak{s}, f : Y \to S^1)$ where $\mathfrak{s}$ is a spin structure on the closed 3-manifold Y, and $(Y_0, \mathfrak{s}_0, f_0) \sim (Y_1, \mathfrak{s}_1, f_1)$ if there exists a compact 4-manifold Z with a spin structure $\mathfrak{t}$ such that $\partial(Z, \mathfrak{t}) = (Y_0, \mathfrak{s}_0) \sqcup (Y_1, \mathfrak{s}_1)$ and
$$\begin{tikzcd} (Y_0, \mathfrak{s}_0) \arrow[d, hook, "\partial_0"'] \arrow[dr, "f_0"] & \\ (Z, \mathfrak{t}) \arrow[r, "\exists F"] & S^1 \\ (Y_1, \mathfrak{s}_1) \arrow[u, hook, "\partial_1"] \arrow[ur, "f_1"'] & \end{tikzcd}$$

Proof of the theorem

Step 1: Build compact, connected, spin 4-manifold V such that
$$\begin{tikzcd} S^3_0(K) \arrow[r, "f"] \arrow[d, hook, "\partial"'] & S^1 \\ V \arrow[ur, "F"'] & \end{tikzcd}$$

Note: F is not a homotopy equivalence.

In the rest of the proof we will upgrade F (and V) until we get a homotopy equivalence.

Recall: if $F_*: \pi_i V \longrightarrow \pi_i(S^1)$ is an isomorphism for all i, then F is a homotopy equivalence by Whitehead's theorem.

For us, by Poincaré-Lefschetz duality, it will be enough to arrange for F_* to be an isomorphism for $i = 0,1,2$.

Proof of the theorem

Step 1: Build compact, connected, spin 4-manifold V such that

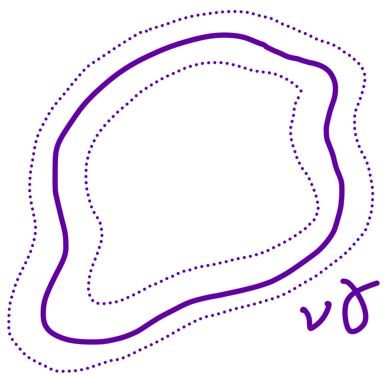
$$\begin{tikzcd} S^3_0(K) \arrow[r, "f"] \arrow[d, hook, "\partial"'] & S^1 \\ V \arrow[ur, "F"'] & \end{tikzcd}$$

Step 2: Arrange for F_* to be an isomorphism on π_1

By construction, F_* is already a surjection on π_1

To make it an injection as well, perform surgery on circles

Given γ in the kernel of F_* replace V by $V \setminus \mathring{\nu}\gamma \cup (S^2 \times D^2)$



Note: $\nu\gamma \cong S^1 \times D^3$, so we have $\partial \nu\gamma = \partial(S^2 \times D^2)$.

There are two choices of gluing map. Use the one so that the result is still spin.

Proof of the theorem

Step 1: Build compact, connected, spin 4-manifold V such that
$$\begin{tikzcd} S^3_0(K) \arrow[r, "f"] \arrow[d, hook, "\partial"'] & S^1 \\ V \arrow[ur, "F"'] & \end{tikzcd}$$

Step 2: Arrange for F_* to be an isomorphism on π_1

Step 3: Replace V with some V' with hyperbolic intersection form

Consider $(\pi_2(V), \lambda_V, \mu_V) \in L_4(\mathbb{Z}[\mathbb{Z}])$, the L-group of nonsingular quadratic forms

Recall: $\pi_2(V) = H_2(V; \mathbb{Z}[\mathbb{Z}])$ since $\pi_1(V) \cong \mathbb{Z}$.

By surgery on circles we can assume $\pi_2(V)$ is a free $\mathbb{Z}[\mathbb{Z}]$-module

Since K has Alexander polynomial one, i.e. $H_1(S^3_0(K); \mathbb{Z}[\mathbb{Z}]) = 0$,

the equivariant intersection form $\lambda_V : \pi_2(V) \times \pi_2(V) \longrightarrow \mathbb{Z}[\mathbb{Z}]$ is non-singular

Proof of the theorem

Step 1: Build compact, connected, spin 4-manifold V such that

$$\begin{tikzcd} S^3_0(K) \arrow[r, "f"] \arrow[d, "\partial"'] & S^1 \\ V \arrow[ur, "F"'] & \end{tikzcd}$$

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Consider $(\pi_2(V), \lambda_V, \mu_V) \in L_4(\mathbb{Z}[\mathbb{Z}])$, the L-group of nonsingular quadratic forms

Fact: $L_4(\mathbb{Z}[\mathbb{Z}]) \cong 8\mathbb{Z}$ given by the signature

Theorem [Freedman]: There exists a closed, spin, simply connected 4-manifold $E8$ with intersection form the $E8$ form with signature 8.

So if $(\pi_2(V), \lambda_V, \mu_V) = 8n \in 8\mathbb{Z} = L_4(\mathbb{Z}[\mathbb{Z}])$, replace V with $V' = V \# -nE8$.

Note V' is spin, and $(\pi_2(V'), \lambda_{V'}, \mu_{V'}) = 0 \in 8\mathbb{Z} \cong L_4(\mathbb{Z}[\mathbb{Z}])$

Proof of the theorem

Step 1: Build compact, connected, spin 4-manifold V such that
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Step 2: Arrange for F_* to be an isomorphism on π_1

Step 3: Replace V with some V' with hyperbolic intersection form

$(\pi_2(V'), \lambda_{V'}, \mu_{V'}) = 0 \in 8\mathbb{Z} \cong L_4(\mathbb{Z}[\mathbb{Z}])$,

so by definition, $(\pi_2(V'), \lambda_{V'}, \mu_{V'})$ is (stably) hyperbolic

In the simplest case, this means $\pi_2(V')$ has a basis of generic immersions $\{f, g : S^2 \looparrowright V\}$ such that

(i) f, g have trivial normal bundle (since V' is spin);

(ii) $\mu_{V'}(f) = 0$; and

(iii) $\lambda_{V'}(f, g) = 1$.

Proof of the theorem

Step 1: Build compact, connected, spin 4-manifold V such that

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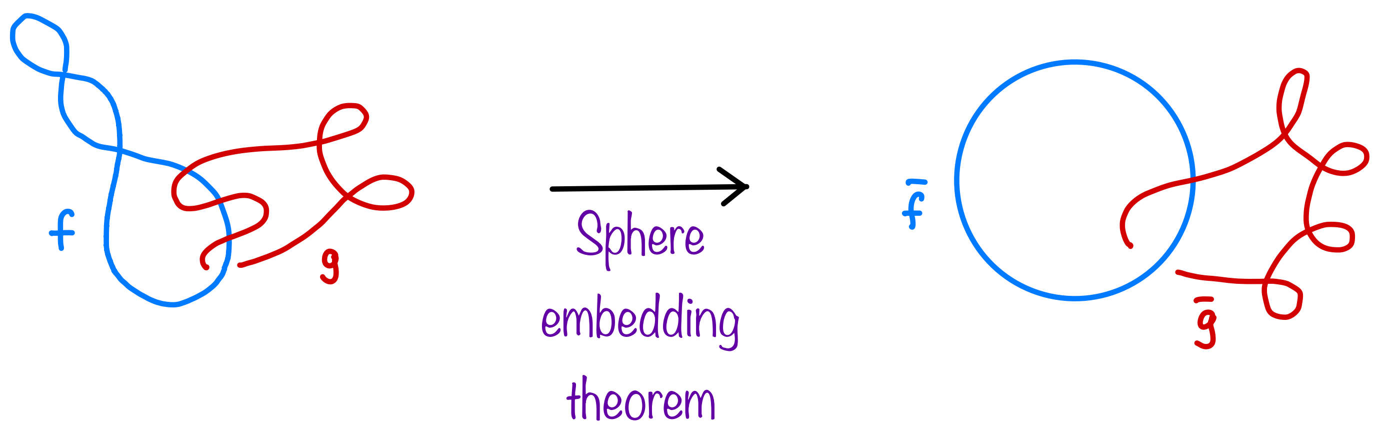
Step 2: Arrange for F_* to be an isomorphism on π_1

Step 3: Replace V with some V' with hyperbolic intersection form

Step 4: Realise half a basis of $\pi_2(V')$ with locally flat, pairwise disjoint, embedded 2-spheres $\{\bar{f}_1, \ldots, \bar{f}_n\}$, equipped with geometrically dual (generically immersed) spheres



Proof of the theorem

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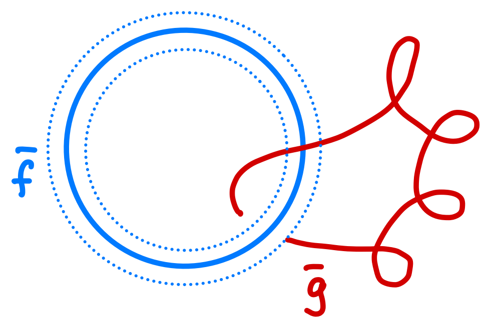
Step 4: Realise half a basis of $\pi_2(V')$ with locally flat, pairwise disjoint, embedded 2-spheres $\{\bar{f}_1, \dots, \bar{f}_n\}$, equipped with geometrically dual (generically immersed) spheres

Step 5: Perform surgery on V' along $\{\bar{f}_1, \dots, \bar{f}_n\}$ and call the result W

i.e. for each i, replace V' by $V' \setminus \mathring{\nu}\bar{f}_i \cup (S^1 \times D^3)$



Check that W satisfies
$$\begin{tikzcd} S^3_0(K) \arrow[r, "f"] \arrow[d, "\partial"'] & S^1 \\ W \arrow[ur, "\simeq"'] & \end{tikzcd}$$

$\square$

Locally flat surfaces in 4-manifolds

Talk 1 (last week):

Theorem [Lee-Wilczyński]: Let M be a closed, simply connected 4-manifold. Then every primitive $\alpha \in H_2(M;\mathbb{Z})$ is represented by a locally flat embedded torus.

Talk 2 (today):

Theorem [Freedman-Quinn]: Let $K \subseteq S^3$ be a knot with Alexander polynomial one. Then K is topologically slice.