Satellite operations and knot concordance

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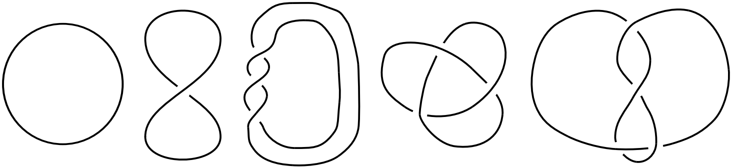
Knots



Definition

A knot is an embedding $S^1 \hookrightarrow \mathbb{R}^3$.

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Two knots are said to be isotopic if one can be deformed into another through embeddings in $\mathbb{R}^3$.

Isotopy is a 3–dimensional equivalence relation.

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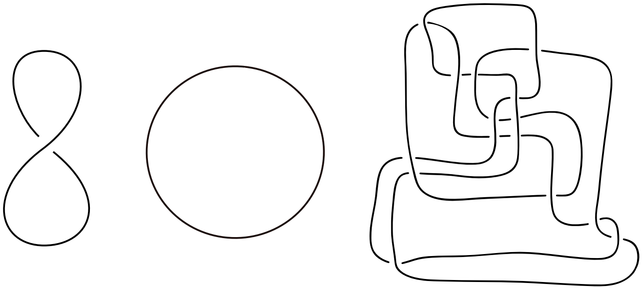

Figure: These are all pictures of the same knot, called the **unknot**.

Knots

Theorem (Lickorish–Wallace, 1960s)

Any closed, connected, orientable manifold can be obtained from $\mathbb{R}^3$ *by performing an operation called 'surgery' on a collection of knots.*

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Knot theory also has applications to algebraic geometry, statistical mechanics, DNA topology, quantum computing, ...

Knot concordance

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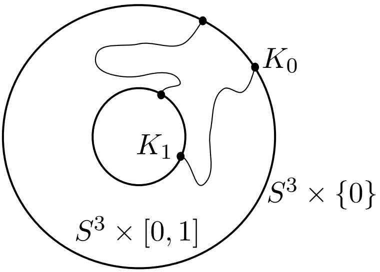
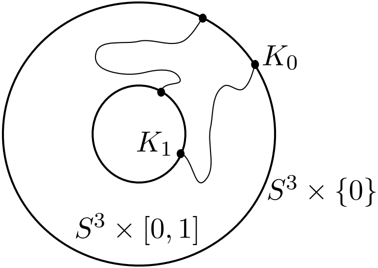
Knots K_0, K_1 are *concordant* if they cobound a smoothly embedded annulus in $S^3 \times [0,1]$. Knots modulo concordance form the *knot concordance group* $\mathcal{C}$.



Topological knot concordance

Definition

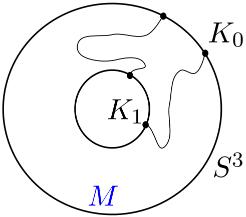
Knots K_0, K_1 are *topologically concordant* if they cobound a locally flat, topologically embedded annulus in $S^3 \times [0,1]$. Knots modulo topological concordance form the *topological knot concordance group* $\mathcal{C}_{\text{top}}$.



Exotic knot concordance

Definition

Knots K_0, K_1 are *exotically concordant* if they cobound a smoothly embedded annulus in a smooth manifold M homeomorphic to $S^3 \times [0,1]$, i.e. a possibly exotic $S^3 \times [0,1]$. Knots modulo exotic concordance form the *exotic knot concordance group* $\mathcal{C}_{\text{ex}}$.



If the smooth 4–dimensional Poincaré Conjecture holds, then $\mathcal{C} = \mathcal{C}_{\text{ex}}$.

Proposition

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If T is the trefoil knot, $g(T) = 1$. Therefore, the trefoil is not equivalent to the unknot.

Connected sum of knots

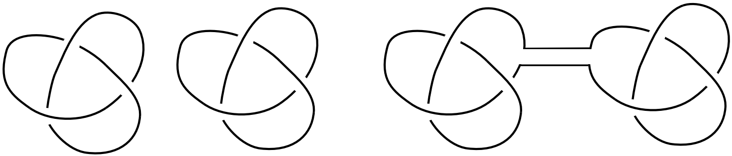

Figure: The connected sum of two trefoil knots, $T\#T$

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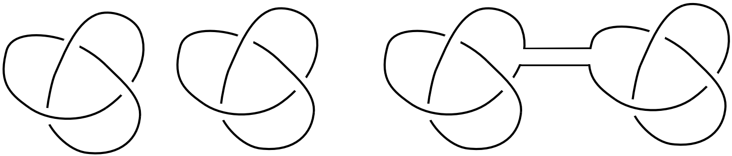

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Corollary: We can never add together non-trivial knots to get a trivial knot.

Slice knots

Recall that a knot is equivalent to the unknot if and only if it is the boundary of a disk in $\mathbb{R}^3$.

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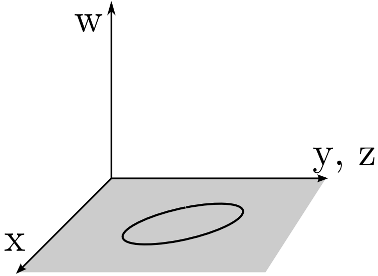
A knot K is *slice* if it is the boundary of a disk in $\mathbb{R}^3 \times [0, \infty)$.



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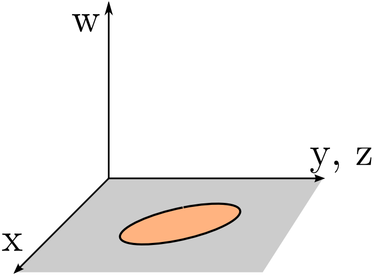

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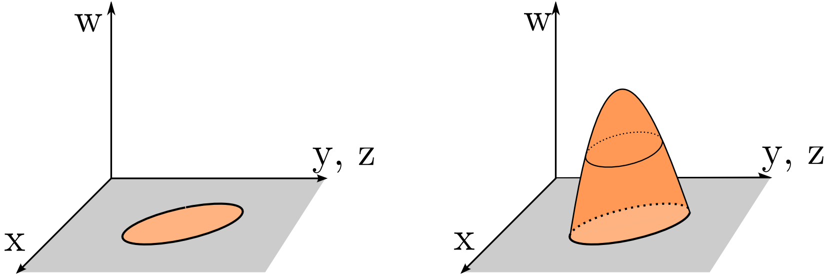
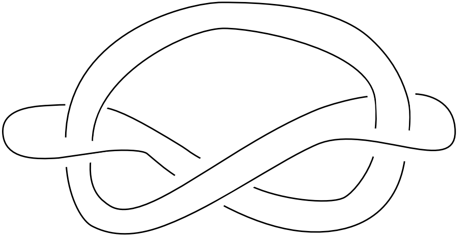


Figure: Schematic picture of the unknot and a slice knot

Examples of slice knots



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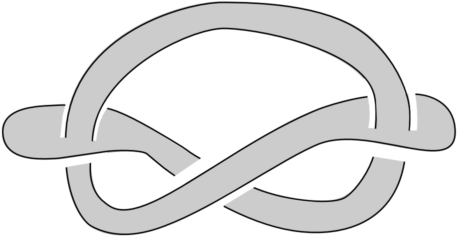

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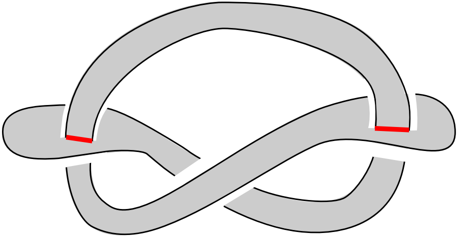

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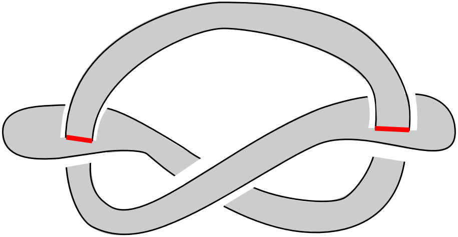

Knots of this form are called *ribbon knots*.

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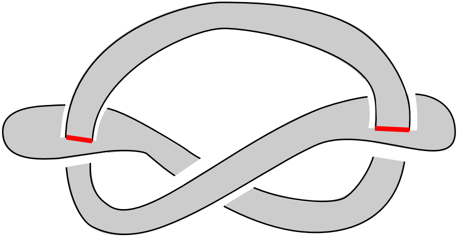

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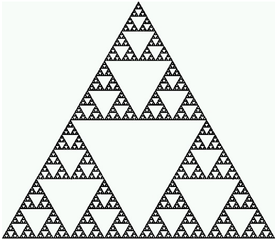
Knots, modulo slice knots, form a group called the *knot concordance group*, denoted $\mathcal{C}$.

Fractals

Fractals are objects that exhibit 'self-similarity' at arbitrarily small scales.



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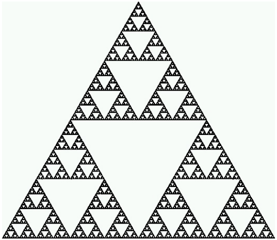

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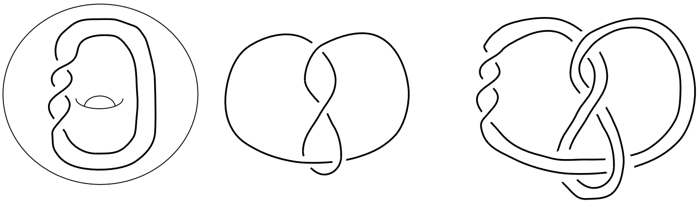
Conjecture (Cochran–Harvey–Leidy, 2011)

The knot concordance group $\mathcal{C}$ is a fractal.

Satellite operations on knots



P K $P(K)$

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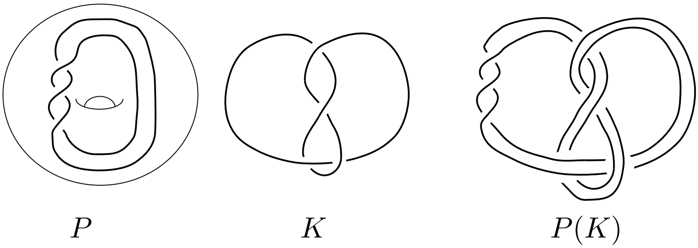

Figure: The satellite operation on knots

Any knot P in a solid torus gives a function on the knot concordance group,

$$P : \mathcal{C} \to \mathcal{C}$$
$$K \mapsto P(K)$$

These functions are called *satellite operators*.

The knot concordance group has fractal properties

Theorem (Cochran–Davis–R., 2012)

Large (infinite) classes of satellite operators $P : \mathcal{C} \to \mathcal{C}$ are injective.

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There are infinitely many satellite operators P *and a large class of knots* K *such that* $P^i(K) \neq P^j(K)$ *for all* $i \neq j$*.*

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Theorem (A. Levine, 2014)

There exist satellite operators that are injective but not surjective.

Fractals

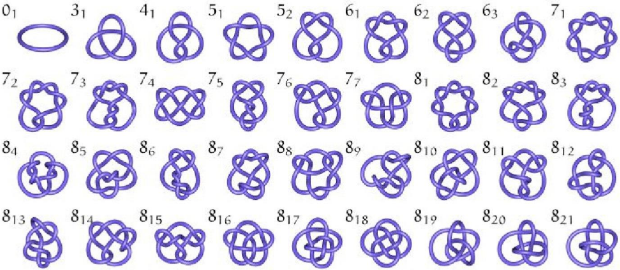
What is left to show?

In order for $\mathcal{C}$ to be a fractal, we need some notion of distance or size, to see that we have smaller and smaller embeddings of $\mathcal{C}$ within itself.

One way to do this is to exhibit a metric space structure on $\mathcal{C}$. There are several natural metrics on $\mathcal{C}$, but we have not yet found one that works well with the current results on satellite operators. The search is on!

The origins of mathematical knot theory

1880s: Kelvin (1824–1907) hypothesized that atoms were 'knotted vortices' in æther. This led Tait (1831–1901) to start tabulating knots.



Tait thought he was making a periodic table!

Examples of knots

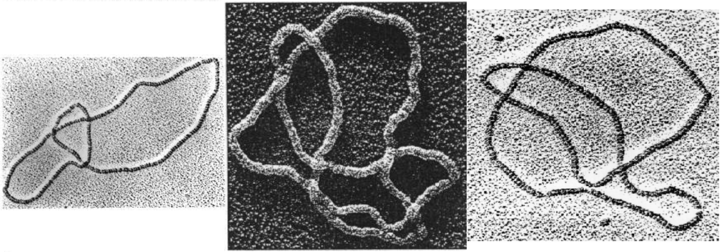


Figure: Knots in circular DNA.

(Images from Cozzarelli, Sumners, Cozzarelli, respectively.)