Satellite operators as group actions on knot concordance

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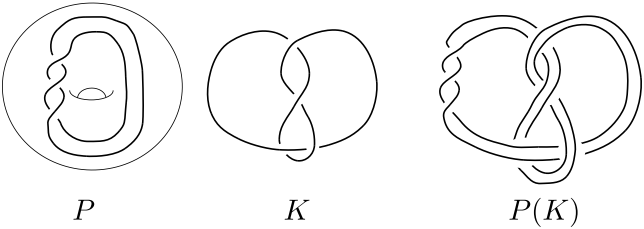
Satellite operators

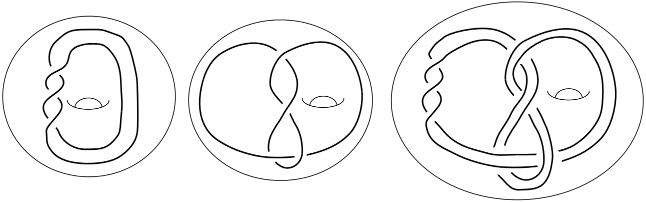
Definition

A *satellite operator* is a knot in the solid torus $S^1 \times D^2$ considered up to isotopy.

Satellite operators act on knots in S^3 via the classical satellite construction.



Satellite operators form a monoid



P $\quad Q \quad P \star Q$

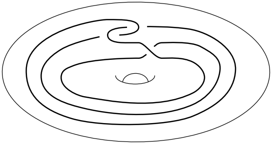
Proposition

The satellite operation gives a monoid action on knots, i.e.

$$(P \star Q)(K) = P(Q(K))$$

Strong winding number one operators

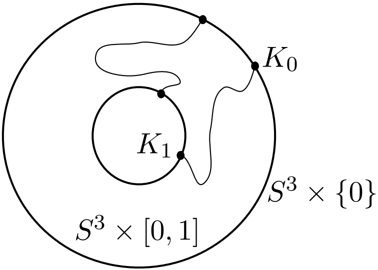
This talk focuses on winding number one satellite operators, particularly so-called *strong winding number one* satellite operators; there exist infinitely many such operators. In particular, any *unknotted* winding number one operator is strong winding number one.



Knot concordance

Definition

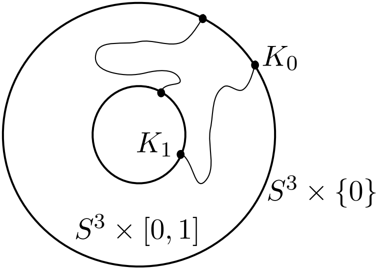
Knots K_0, K_1 are *concordant* if they cobound a smoothly embedded annulus in $S^3 \times [0,1]$. Knots modulo concordance form the *knot concordance group* $\mathcal{C}$.



Topological knot concordance

Definition

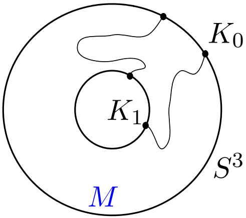
Knots K_0, K_1 are *topologically concordant* if they cobound a locally flat, topologically embedded annulus in $S^3 \times [0,1]$. Knots modulo topological concordance form the *topological knot concordance group* $\mathcal{C}_{\text{top}}$.



Exotic knot concordance

Definition

Knots K_0, K_1 are *exotically concordant* if they cobound a smoothly embedded annulus in a smooth manifold M homeomorphic to $S^3 \times [0,1]$, i.e. a possibly exotic $S^3 \times [0,1]$. Knots modulo exotic concordance form the *exotic knot concordance group* $\mathcal{C}_{\text{ex}}$.



If the smooth 4–dimensional Poincaré Conjecture holds, then $\mathcal{C} = \mathcal{C}_{\text{ex}}$.

Satellite operators act on knot concordance classes

The classical satellite construction descends to a well-defined function on knot concordance classes, i.e. if K and J are concordant, then $P(K)$ and $P(J)$ are concordant, for any P.

Question

What can we say about the action of satellite operators on knot concordance classes?

- Do they act by injections? i.e. for a given operator P, if $P(K) = P(J)$ does it imply that $K{=}J$?

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Theorem (Cochran–Davis–R., 2012)

Any strong winding number one satellite operator gives an injective function on $\mathcal{C}_{top}$ and $\mathcal{C}_{ex}$ (and therefore, modulo the smooth 4–dimensional Poincaré Conjecture, on $\mathcal{C}$).

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Any strong winding number one satellite operator gives an injective function on $\mathcal{C}_{top}$ and $\mathcal{C}_{ex}$ (and therefore, modulo the smooth 4–dimensional Poincaré Conjecture, on $\mathcal{C}$).

- Do they act by surjections? i.e. for a given operator P and knot J, is there a K such that $P(K) = J$?

Goal

We show that satellite operators are (naturally) a subset of a *group*, $\widehat{\mathcal{S}}$.

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We show that satellite operators are (naturally) a subset of a *group*, $\widehat{\mathcal{S}}$. This group acts on concordance classes of knots in homology 3–spheres in a manner that is compatible with the classical satellite construction.

This observation allows us to give a new (easier) proof of the Cochran–Davis-R. result about injectivity, and gives a new approach to the question of surjectivity.

Main theorem

Theorem (Davis–R.)

Let $\mathcal{S}$ be the monoid of strong winding number one satellite operators. Let $\widehat{\mathcal{C}_{top}}$ and $\widehat{\mathcal{C}_{ex}}$ be the groups of topological and exotic concordance classes of knots in homology 3–spheres.

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There exist homomorphisms $E : \mathcal{S} \to \widehat{\mathcal{S}}$, $\Psi : \mathcal{C}_* \hookrightarrow \widehat{\mathcal{C}}_*$ such that the following diagrams commute for each $P \in \mathcal{S}$.

$$\begin{array}{ccc} \mathcal{C}_{ex} & \xrightarrow{P} & \mathcal{C}_{ex} \\ \downarrow \Psi & & \downarrow \Psi \\ \widehat{\mathcal{C}}_{ex} & \xrightarrow{E(P)} & \widehat{\mathcal{C}}_{ex} \end{array} \qquad \begin{array}{ccc} \mathcal{C}_{top} & \xrightarrow{P} & \mathcal{C}_{top} \\ \downarrow \Psi & & \downarrow \Psi \\ \widehat{\mathcal{C}}_{top} & \xrightarrow{E(P)} & \widehat{\mathcal{C}}_{top} \end{array}$$

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Since $E(P)$ is a group element, it acts on $\widehat{\mathcal{C}_*}$ by a bijection. The Cochran–Davis–R. result follows.

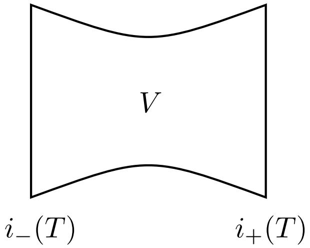
Homology cylinders

Let T be the torus $S^1 \times S^1$. A *homology cylinder* on T is a triple (V, i_+, i_-) where

- V is a compact, connected, oriented 3–manifold
- For $\epsilon = \pm 1$, $i_\epsilon : T \to \partial V$ is an embedding
- i_+ is orientation-preserving and i_- is orientation-reversing
- $\partial V = i_+(T) \sqcup i_-(T)$
- $(i_\epsilon)_* : H_*(T) \to H_*(V)$ is an isomorphism

A homology cylinder (V, i_+, i_-) is called a *strong cylinder* if $\pi_1(V)$ is normally generated by each of $\text{Im}(i_+)_*$ and $\text{Im}(i_-)_*$.

Homology cylinders form a group



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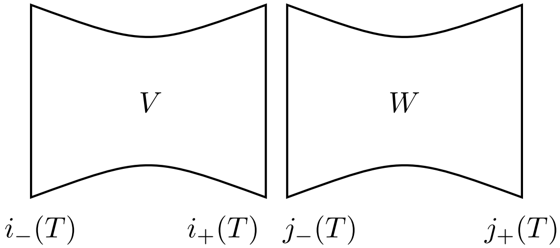

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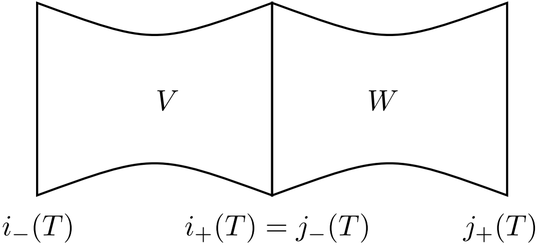
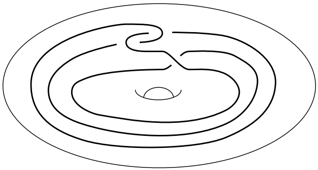


Stacking gives a monoid operation on homology cylinders. Under homology cobordism, homology cylinders form a group (Levine).

Satellite operators yield homology cylinders



Given a satellite operator P in a solid torus V, carve out a neighborhood of P inside V. The resulting 3–manifold has two toral boundary components, with canonical maps to the torus $T = S^1 \times S^1$.

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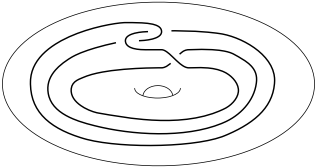

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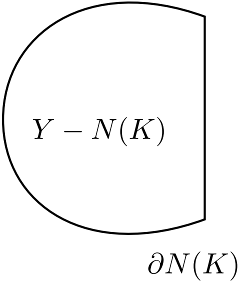
A strong winding number one satellite operator yields a strong homology cylinder.

Homology cylinders act on knots in homology 3–spheres

Given a knot K in a homology 3–sphere Y, carve out $N(K)$, a solid torus neighborhood of K.



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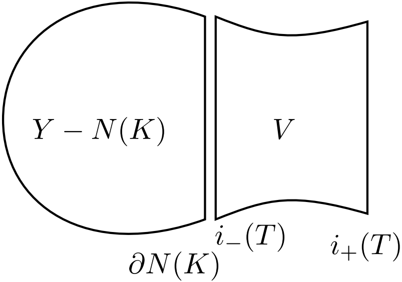

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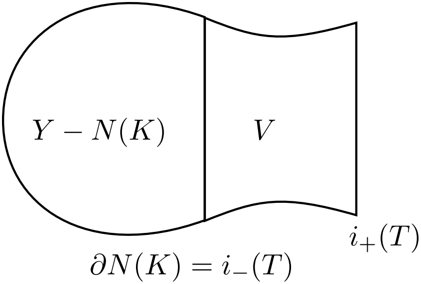

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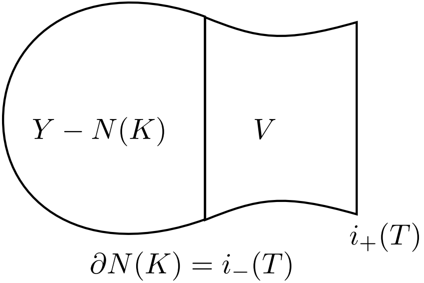

We obtain a 3–manifold with a single torus boundary component. We can canonically glue in a solid torus to get a homology 3–sphere. The core of this solid torus is the new knot.

Surjectivity of satellite operators

For each strong winding number one satellite operator P, the following diagram commutes.

$$\begin{array}{ccc} \mathcal{C}_* & \xrightarrow{P} & \mathcal{C}_* \\ \Big\downarrow{\scriptstyle\Psi} & & \Big\downarrow{\scriptstyle\Psi} \\ \widehat{\mathcal{C}}_* & \xrightarrow{E(P)} & \widehat{\mathcal{C}}_* \end{array}$$

Since $E(P)$ is an element of the group $\widehat{\mathcal{S}}$, it has an inverse $E(P)^{-1}$.

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Since $E(P)$ is an element of the group $\widehat{\mathcal{S}}$, it has an inverse $E(P)^{-1}$.
If $E(P)^{-1}(\mathcal{C}_*) \subseteq \mathcal{C}_*$ then P is surjective on $\mathcal{C}_*$.
The following is an example of a bijective satellite operator.