SUPERSLICE KNOTS HAVE ALEXANDER POLYNOMIAL ONE

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A knot K is said to be (topologically) *superslice* if it bounds a flat, properly embedded disc Δ in B^4 such that the double of (B^4, Δ) is isotopic to (S^4, U), where U is the unknotted 2-sphere. In [LM15, p. 1019], it is asserted that any superslice knot has Alexander polynomial one, with a reference to a paper of Gordon-Sumners [GS75]. However, we have not been able to detect a proof of this fact within the latter paper. Jeffre Meier later pointed us to a proof in [Rub16, Corollary 1.3], but by then we had already found the elementary proof we now give.

Proposition 1. *If K is a superslice knot with superslicing disc Δ then $\pi_1(B^4 \setminus \Delta) \cong \mathbb{Z}$ and consequently, $\Delta_K(t) \doteq 1$.*

Proof. Let Δ be a superslicing disc for K and Σ denote the double of Δ in S^4. By definition, Σ is unknotted and thus, $\pi_1(S^4 \setminus \Sigma) \cong \mathbb{Z}$. We also know that $S^4 \setminus \Sigma$ is the double of $B^4 \setminus \Delta$ and thus there is a retraction $r\colon S^4 \setminus \Sigma \to B^4 \setminus \Delta$, which is sometimes called the *folding map.* In other words,

$$r \circ \iota\colon B^4 \setminus \Delta \to B^4 \setminus \Delta,$$

where ι denotes inclusion, is the identity map. Consequently, the map r_* on fundamental groups is surjective. As a result, $\pi_1(B^4 \setminus \Delta)$ is a quotient of $\pi_1(S^4 \setminus \Sigma) \cong \mathbb{Z}$. Since $H_1(B^4 \setminus \Delta) \cong \mathbb{Z}$, we conclude that $\pi_1(B^4 \setminus \Delta) \cong \mathbb{Z}$. The latter implies that $\Delta_K(t) \doteq 1$. □

The above establishes the following theorem.

Theorem 2. *A knot K is (topologically) superslice if and only if $\Delta_K(t) \doteq 1$. Indeed, a slice disc Δ for a knot is a superslicing disc if and only if $\pi_1(B^4 \setminus \Delta) \cong \mathbb{Z}$.*

Proof. Proposition 1 establishes that if Δ is a superslicing disc for K then $\pi_1(B^4 \setminus \Delta) \cong \mathbb{Z}$ and $\Delta_K(t) \doteq 1$. We now show the converse. Let Δ be a slice disc for K with $\pi_1(B^4 \setminus \Delta) \cong \mathbb{Z}$. The existence of such a disc is equivalent to the condition that $\Delta_K(t) \doteq 1$. Let Σ be the double of Δ in S^4. By the Seifert-van Kampen theorem, $\pi_1(S^4 \setminus \Sigma) \cong \mathbb{Z}$. By [FQ90, Theorem 11.7A], this implies that Σ is (topologically) isotopic to the unknot as desired. □

In contrast, in the smooth category, it was shown by Ruberman [Rub16] that there exist knots with Alexander polynomial one which are not smoothly superslice.

References

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