Class 20

Dec 18.
Tue

§ One more example: $\mathbb{C}P(n)$

$$\mathbb{C}P(n) = \mathbb{C}^{n+1}\setminus\{0\} \big/ (z_0 .. z_n) \sim \lambda(z_0 .. z_n),\ \lambda \in \mathbb{C}\setminus\{0\}$$

$$= \{ [z_0 : z_1 : \cdots : z_n] \mid z_i \in \mathbb{C} \text{ not all zero, normalize s.t. } \max_i |z_i| = 1 \}$$

$$\begin{aligned} \psi_i : \mathbb{C}^n &\longrightarrow \mathbb{C}P(n) \\ (z_1 .. z_n) &\longmapsto [z_1 : \ldots : z_i : 1 : z_{i+1} : \ldots : z_n] \end{aligned}$$

$D \subseteq \mathbb{C}$ unit disc, $\quad B_i = \psi_i(D \times \ldots \times D)$

This gives a handle decomposition for $\mathbb{C}P(n)$:

$$p \in B_i \iff |z_i| = 1$$
$$p \in \operatorname{int}(B_i) \iff |z_j| < 1 \quad \forall j \neq i$$

$\Rightarrow$ $\{B_i\}$ cover $\mathbb{C}P(n)$,
$\operatorname{int} B_i \cap \operatorname{int} B_j = \emptyset$
so they can only intersect along parts of their boundary.

<u>Claim.</u> B_k intersects $\bigcup_{i<k} B_i$ along $\psi_k(\partial(\underbrace{D \times .. \times D}_{k}) \times \underbrace{D \times .. \times D}_{n-k})$

$\Longrightarrow$ B_k is attached to $\bigcup_{i<k} B_i$ as a $2k$-handle.

$\mathbb{C}P(2) = B_0 \cup B_1 \cup B_2$

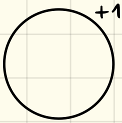
$p \in B_0 \cap B_1 \qquad p = \psi_0(w_1, w_2) = [1 : w_1 : w_2]$
$\qquad = \psi_1(z_1, z_2) = [z_1 : 1 : z_2] \quad \Rightarrow \quad z_1 \neq 0$

$\Rightarrow \quad w_1 = z_1^{-1}$
$\qquad w_2 = z_2 z_1^{-1}$

Attaching map: $\partial D \times D \longrightarrow \partial D \times D$

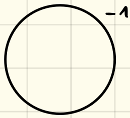
$(z_1, z_2) \longmapsto (z_1^{-1}, z_2 z_1^{-1})$

Get Kirby diagrams:



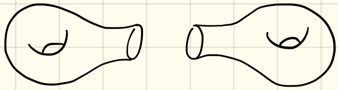
$\mathbb{C}P^2$

and



$\overline{\mathbb{C}P^2}$

Recall: connected sums of oriented manifolds

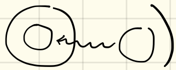


(for smooth manifolds:

can isotope two different choices to each other

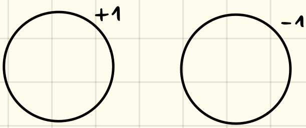
for topological 4-manifolds:

annulus thm (Quinn, following Freedman)



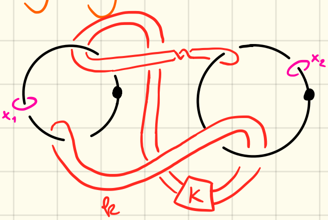
)

Diagram for $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$



§ Can use Kirby diagrams to compute π_1, H_*, H^*

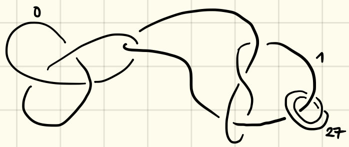
π_1:



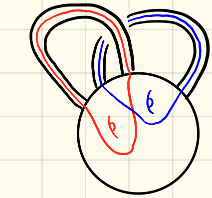
gen. $x_1, x_2 \ldots$ (for 1-handles)

rel. given by words in x_i (2-handles)

Suppose we have a diagram without 1-handles

$X_1 :=$ 

schematically:



If I have 1-handles, need to see which curves are null-homologous in $\partial(\text{0-h} \cup \text{1-handles})$.

§ Intersection forms.

X compact oriented (topological) 4-manifold
$[X]$ fundamental class

$$Q_X : H^2(X, \partial X; \mathbb{Z}) \times H^2(X, \partial X; \mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$(a, b) \longmapsto (a \cup b)[X]$$

or

$$Q_X : H_2(X; \mathbb{Z}) \times H_2(X; \mathbb{Z}) \longrightarrow \mathbb{Z}$$

count inter. points with sign

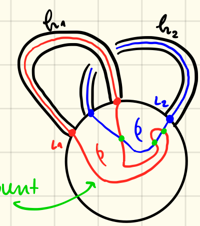
FACT: Each element of $H_2(X;\mathbb{Z})$ repr. by emb. oriented surfaces.

Claim. Q_X for a 4-mfld with no 1- or 3-handles is just the linking-framing matrix for the diagram:

example.

$$X_1 \rightsquigarrow \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & 2 \\ 0 & 2 & 27 \end{bmatrix}$$

Just recall how we defined linking numbers for a link!



algebraic count $= lk(L_1, L_2)$

Theorem [Milnor-Whitehead]

Any 2 simply-conn. oriented 4-mflds are homotopy equivalent iff they have isomorphic inter. forms.

Theorem [Freedman]

- Given an even unimodular symmetric bilinear form, $\exists$ unique closed simply-conn. oriented 4-mfld realizing it as its Q_X.
- Given an odd unimodular bilinear form, $\exists$ two such 4-mflds up to homeomorphism. and at most one is smooth.

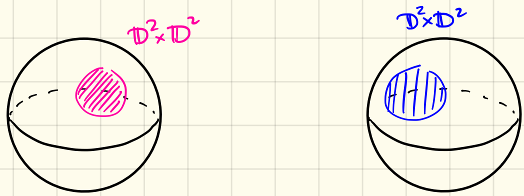
Def. unimodular if has $\det = \pm 1$. (if $\partial X = \emptyset$, Q_X is unimodular)

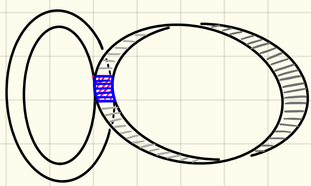
A bilinear form is even if $Q(x,x)$ is even for all x

-||- is odd otherwise.

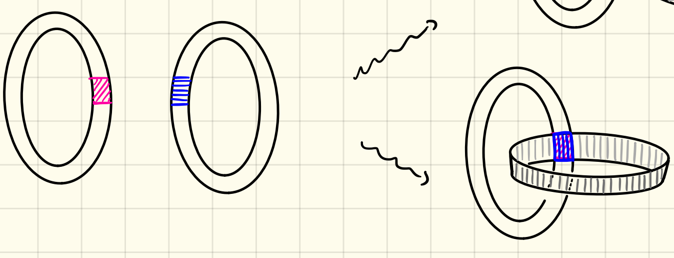
Def. M closed 4-mfld. $\sigma(M) := $ signature of Q_M.

§ PLUMBING.

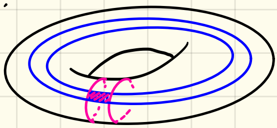




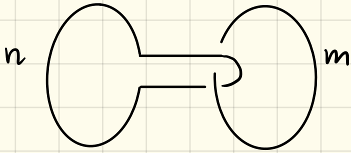
lower dimension: $S^1 \times D^1$ $S^1 \times D^1$



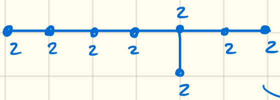
Note: case 2 is a manifold! It looks like nbhd of meridian + longitude on a torus:



So plumbing of 2 disk bundles over S^2 will have Kirby diagram:

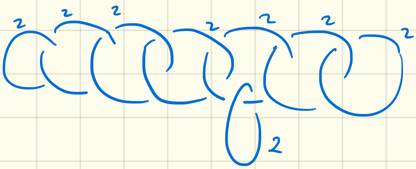


Example. Take the E_8-form
unimodular & even



$$\begin{bmatrix} 2 & 1 & & & \\ 1 & 2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & \\ & & & & 2 \end{bmatrix}$$

can be represented by

$X :=$ 

$H_*(\partial X; \mathbb{Z}) \cong H_*(S^3; \mathbb{Z})$ (∂X is a integer homology 3-sphere)

Freedman: Any $\mathbb{Z}$-homology sphere bounds a contractible 4-mfld.
only known to be topological.

Then: glue this W to X to get a 4-mfld with $\sigma = 8$.
But:
Rokhlin: The signature of a smooth closed simply-conn. 4mfld with even int. form is divisible by 16.

$\Rightarrow$ $\nexists$ closed simply-conn. smooth 4-mfld with E_8 as its inter. form.

Donaldson: X smooth simpl. conn. closed, Q_X positive definite then $Q_X = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$.

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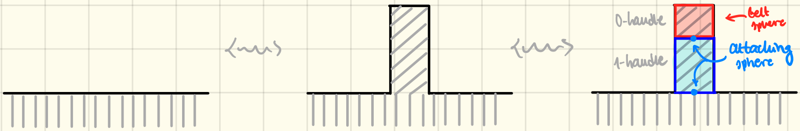
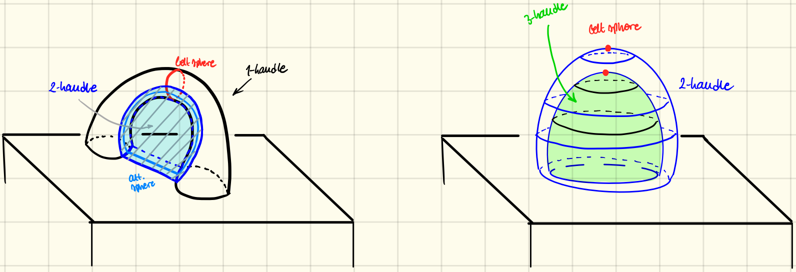
Today: handle cancellation
handle slides
h-cobordism thm & Poincaré conjecture (dim ≥ 5)

§ HANDLE BIRTH / DEATH

2-dim :

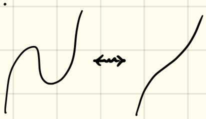


3-dim :

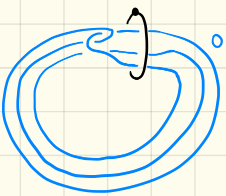


Morse Cancellation Lemma.
A $(k-1)$-handle $h^{(k-1)}$ can be cancelled by a k-handle $h^{(k)}$ if the attaching sphere of $h^{(k)}$ intersects the belt sphere of $h^{(k-1)}$ transversely in a single point.

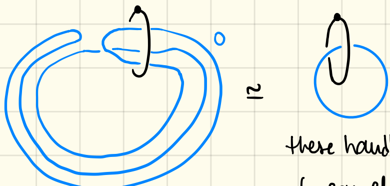
idea:



example.

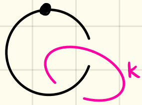


algebraic inter. number = 1
they do not cancel!
(not obvious)

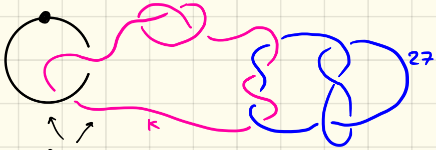


these handles do cancel!

Note: framing of the 2-handles and interaction with other handles does not matter:

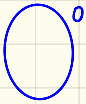


can cancel!



can cancel this 1/2-handle pair

Cancelling 2-/3-handle pair:

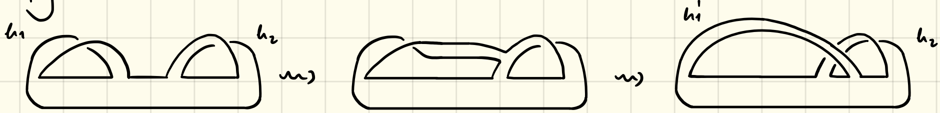


∪ (3-handle) not drawn

Theorem [Cerf] Any two handle decompositions of the same space are related by isotopies and handle creation/cancellation.

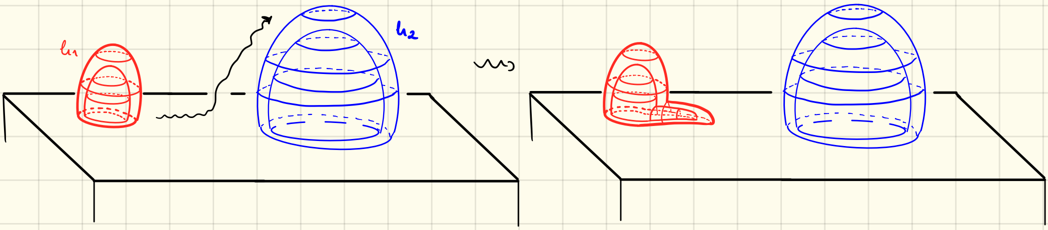
a particular type of isotopy are HANDLE SLIDES

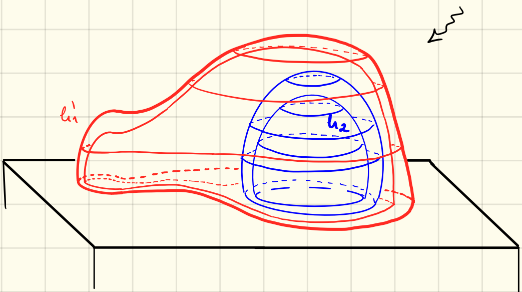
2-dim



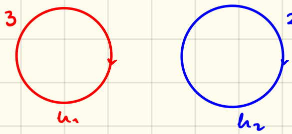
3-dim Sliding 2-handles:

framing and att. sphere of h_1 changed, h_2 did not move

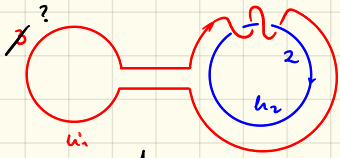




4-dim. What is the att. circle of h_1' ?



slide $\rightsquigarrow$



this band can
- twist
- knot
- link other 2-handles and 1-handles

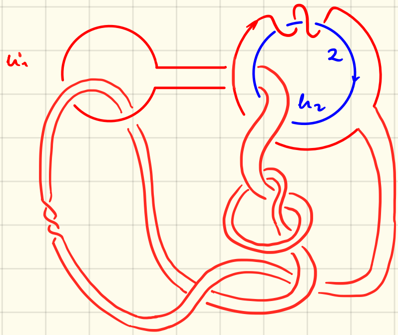
What is the framing of h_1' ?

$\{\alpha_1 \dots \alpha_m\}$ basis of $H_2(X \cup \{h_i\}_{i=1}^m)$

$\Rightarrow$ handle slide of h_i over h_j leads to:

$$\begin{cases} \alpha_i' = \alpha_i \pm \alpha_j \\ \alpha_k' = \alpha_k \quad \text{for } k \neq i \end{cases}$$

e.g.



the band follows the guiding arc for the handle slide.

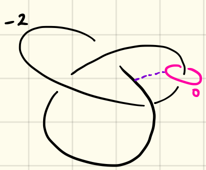
Recall: if no 1-handles the inter. form $Q_{D^4 \cup \{h_i\}}$ is given by the linking-framing matrix, so:

$$(\alpha_i \pm \alpha_j)^2 = \alpha_i^2 + \alpha_j^2 \pm 2 lk(h_i, h_j)$$

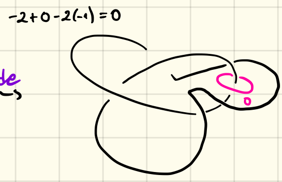
linking number of att. circles

$$\Rightarrow \quad n_i' = n_i + n_j \pm 2\, lk(h_i, h_j)$$

Example



slide $\rightsquigarrow$

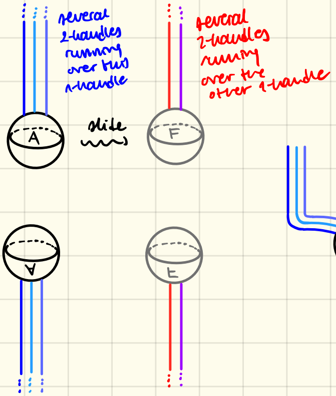


isotopy $\rightsquigarrow$

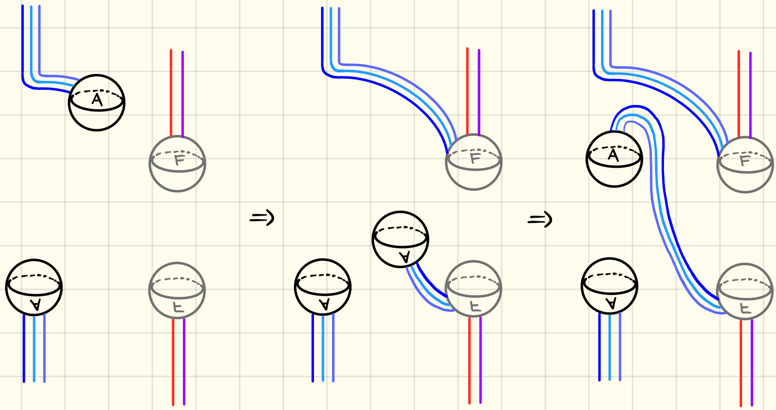
this is $S^2 \times S^2$ (or missing a ball if no 4-handle)

§ Sliding a 1-handle over another 1-handle

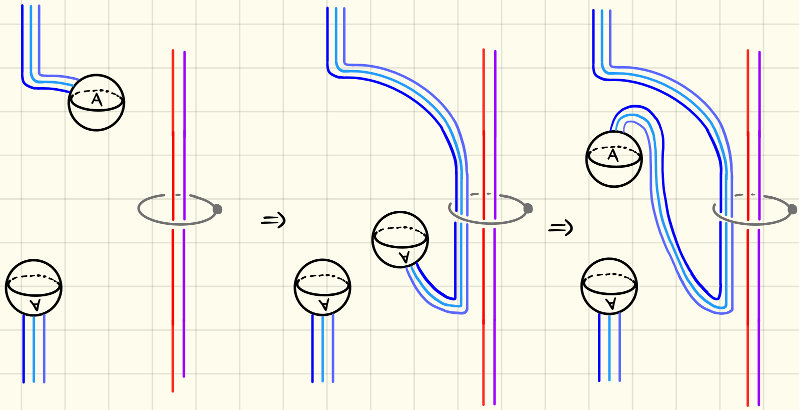
IN OLD NOTATION:



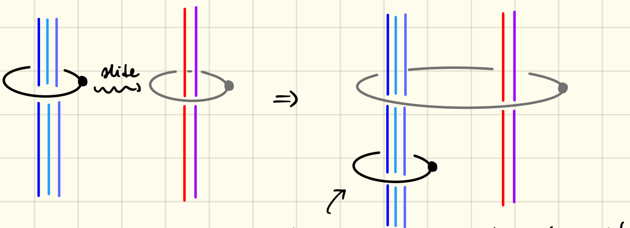
Step-by-step:



"HYBRID" NOTATION

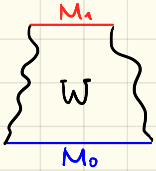


NEW NOTATION



this is the 1-handle doing the slide!

§ H-COBORDISMS



W^{d+1} is an h-cobordism if:

- $\partial W = -M_0 \sqcup M_1$
- $M_i \hookrightarrow W$ is a homotopy equivalence for $i=0,1$

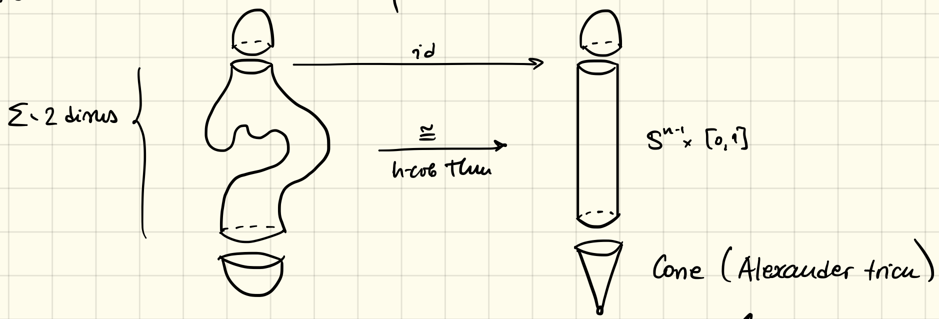
h-cobordism thm (Smale '60s):

Any smooth simply-connected h-cobordism W^{d+1} is diffeomorphic to the product $M_0 \times [0,1]$ if $d \geq 5$.

Poincaré Conjecture: Any smooth homotopy n-sphere Σ^n $n \geq 5$ is homeomorphic to S^n.

proof.

For $n \geq 6$: Remove two discs from Σ^n.



note: we lose differentiability here.

For $n=5$: Use $\Sigma^5 = \partial V^6$ with V^6 contractible.
Take out a ball from V^6.
Get an h-cob. of dim $d+1=6$ $\Rightarrow$ $\Sigma^5 \cong S^5$. ▨

sketch of proof of h-cobordism thm. W smooth $\Rightarrow$ $\exists$ relative handle decomp wrt. M_0

- Can assume no 0-handles
- "handle trading": can replace 1-handles by 3-handles (uses simply-connectedness)
- note: $H_*(W, M_0) = 0$ $\Rightarrow$ all handles cancel algebraically. Pair them up using handle slides.

i.e. use handle slides to realize a basis change until each handle either cancels or is cancelled.

Suppose for example h^3 cancels algebraically h^2.
$\Leftrightarrow$ attaching sphere of h^3 intersects belt sphere of h^2 alg. once.

Use Whitney trick to obtain geometrically once. Can cancel them!

$\Rightarrow$ Cobordism without handles is a cylinder! $\boxtimes$

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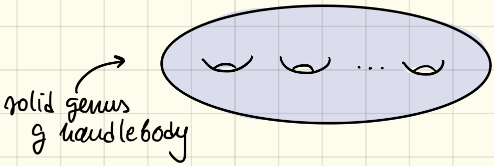
Class 22

§ 3-MANIFOLDS.

Jan 8 TUE

Recall:

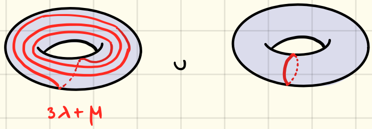
We saw that any closed 3-manifold has a Heegaard decomposition:



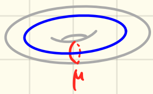
$M^3 = H_g \cup_\varphi H_g$
$\varphi : \partial H_g \longrightarrow \partial H_g$
for some genus g

Note: μ = meridian
λ = longitude

Example.

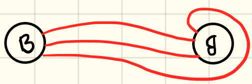


is $L(3,1)$



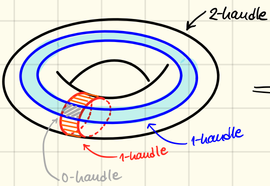
Heegaard splittings can be given by diagrams in the plane:
= blackboard
= $\mathbb{R}^2$ ($\cup \infty$)

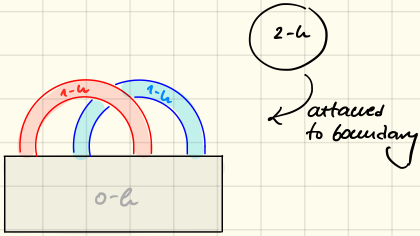
$L(3,1)$:



How to draw $\mathbb{T}^3 = S^1 \times S^1 \times S^1$?

Step 1) handle decomposition of 2-torus

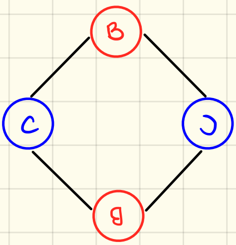
$\mathbb{T}^2 =$ 

$=$ 

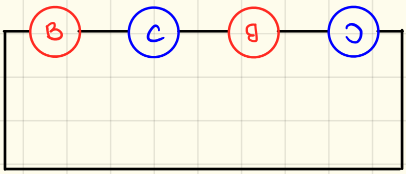
Step 2) handle decomposition of $\mathbb{T}^2 \times I$ 2-dim' $\begin{cases} 0\text{-h} \\ 1\text{-h} \\ 2\text{-h} \end{cases}$ $\Rightarrow$ 3-dim' $\begin{cases} 0\text{-h} \\ 1\text{-h} \\ 2\text{-h} \end{cases}$

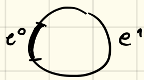
so:

have two 1-handles and one 2-handle attached to analogous curve as in picture above:



2-handle

$=$ 

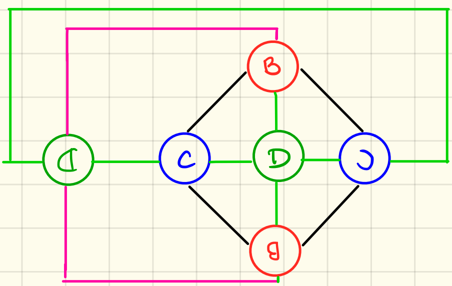


Step 3) $\mathbb{T}^3 = \mathbb{T}^2 \times S^1 = \underbrace{(h^0 \cup h^1_a \cup h^1_b \cup h^2)}_{\text{these give } \mathbb{T}^2 \times I} \times (e^0 \cup e^1)$ add this now

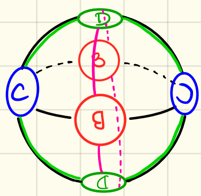
* $h^0 \times e^1$ is a 1-handle with core $=$ core$(h_0) \times$ core(e^1)

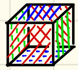
  and att. sph $= \partial$ core $=$ a.s( ) $\times$ core(e^1)

* a.s$(h^1_a \times e^1) =$ a.s.$(h^1_a) \times$ core(e^1) $\cup$ core$(h^1_a) \times$ a.s.e^1 $\cup$ core$(h^0) \times$ a.s.(e^1)

$\Rightarrow \mathbb{T}^3 =$ 

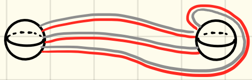
More symmetric picture on S^2:



Aside: can think of $\mathbb{T}^3$ as a cube with oposite faces identified: 

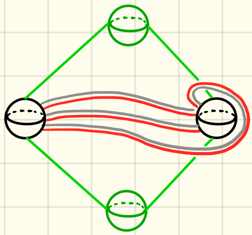
How to draw $L(3,1) \times S^1$?

$L(3,1) \times I$



2-handle with blackboard framing ← in grey

$L(3,1) \times S^1$

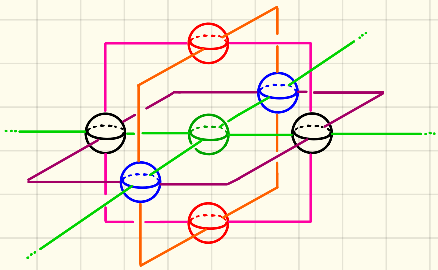


we add one 1-h. and one 2-h

How to draw $\mathbb{T}^4$?

again: take $\mathbb{T}^3 \times I$ and add one 1-h and two 2-h.
(= thicken picture for $\mathbb{T}^3$)

$\mathbb{T}^4$



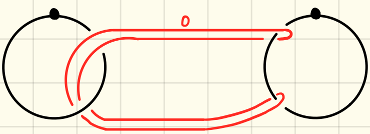
the other green  is at ∞.

all 2-handles are 0-framed.

[see Gompf-Stipsicz fig 4.42 p137]

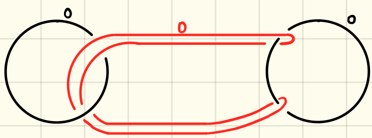
How to draw $\mathbb{T}^2 \times \mathbb{D}^2$?

$\mathbb{T}^2 \times \mathbb{D}^2$

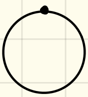
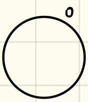


$\partial = \mathbb{T}^3$

Note:



also has $\partial = \mathbb{T}^3$

Note: $\partial($  $) = \partial($  $) = S^1 \times \mathbb{D}^2$ (see HW A2)

§ SURGERY or "spherical modification":

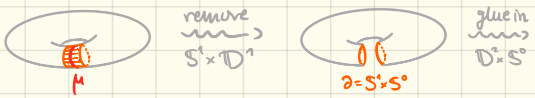
Let $\operatorname{int} M^n \supset S^k \times D^{n-k}$ (M can have non-empty boundary)

Surgery is a procedure:

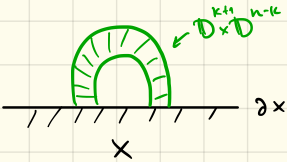
$$M \rightsquigarrow \underbrace{(M \setminus S^k \times D^{n-k})}_{\substack{\text{has "new"} \\ \text{boundary component} \\ S^k \times S^{n-k}}} \cup (D^{k+1} \times S^{n-k-1})$$

glue so that $D^{k+1} \times \{*\}$ is bounded by $S^k \times \{*\}$

Example of surgery on T^2:



! ATTACHING HANDLES changes boundary by surgery:



schematic of att. of a $(k+1)$-handle to the $(n+1)$-dim'l mfld X

$$\partial (D^{k+1} \times D^{n-k}) = (S^k \times D^{n-k}) \underset{S^k \times S^{n-k-1}}{\cup} (D^{k+1} \times S^{n-k-1})$$

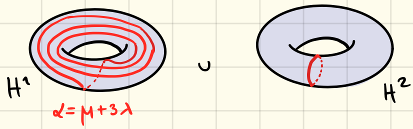
$$\partial (X \cup D^{k+1} \times D^{n-k}) = \text{surgery on } S^k \times D^{n-k} \text{ in } \partial X.$$

§ DEHN SURGERY.

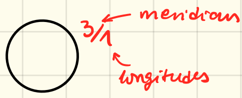
Let $K \subset S^3$ with a tubular nbhd νK

Then

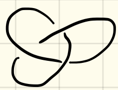
$$(S^3 \setminus \nu K) \cup (D^2 \times S^1)$$

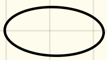
gluing map given by any simple closed curve on the new boundary torus e.g. 

Can think of this Heeg. decomp of $L(3,1)$ also as a Dehn surgery.
Namely: $\quad H^1 = S^3 \setminus \nu U$ and we are gluing H^2 using α
On one hand: $\alpha = \mu + 3\lambda$ on $\partial H^1 = \mathbb{T}^2$
but on the other hand: $\alpha = \lambda + 3\mu$ on $\partial(\nu U) = \mathbb{T}^2$
The latter is Dehn surgery on $(3,1)$-curve on U, we write:

U 

"Dehn surgery diagram"

e.g.  $1/2$ is a $\mathbb{Z}HS^3$.

e.g.  p/q is the lens space $L(p,q)$

Fundamental Theorem of 3-manifolds. [Kirby]

$$\left\{ \begin{array}{c} \text{framed or. links in } S^3 \\ \text{(unknotted components,} \\ \text{framings } = \pm 1) \end{array} \right\} \Big/ \begin{array}{c} \text{isotopy} \\ + \text{ blow up/down} \\ (+ \text{ handle slides}) \end{array} \xrightarrow[1:1]{} \left\{ \begin{array}{c} \text{(connected) closed} \\ \text{oriented} \\ \text{3-manifolds} \end{array} \right\} \Big/ \text{homeo}$$

//

Class 23 Jan 10 THU

§ Surgery & Dehn Surgery

Recall:

Surgery is the effect on the boundary when we attach a handle.

i.e.

$$\partial^+ (X^{n+1} \cup_{\hat{\varphi}} h) = \text{result of doing surgery on } \partial^+ X = M^n \text{ along the attaching sphere}$$

where $\hat{\varphi} : S^k \times D^{n-k} \hookrightarrow M^n$ is the att. map for h

or equivalently:

$\varphi : S^k \hookrightarrow M^n$ (att. sphere)

with framing f of the normal bundle

So surgery on (φ, f) is the manifold:

$$\left(M^n \setminus \hat{\varphi}(S^k \times D^{n-k}) \right) \cup_{\hat{\varphi}|_{S^k \times S^{n-k-1}}} \left(D^{k+1} \times S^{n-k-1} \right)$$

Isotopy class of (φ, f) determines the result up to diffeo.

Prop. If $C \subset M^4$ is a null-homotopic circle, then the result of surgery on M along C is either:

$$M \# S^2 \times S^2 \quad \text{or} \quad M \# S^2 \tilde{\times} S^2$$

(these might not be distinct, depending on M)

(recall that there are precisely two S^2-bundles over S^2, clutching function corresponds to an elt of $\pi_1 SO(3) \cong \mathbb{Z}/2$)

proof. Write $M = M \# S^4$

Consider $C_0 \subset S^3 \subset S^4$ with C_0 null-homotopic.

In particular, C is homotopic to C_0.

Homotopy implies isotopy for loops in a 4-manifold

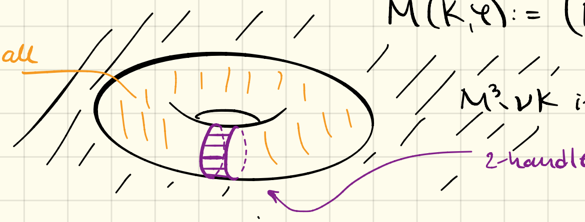
$\Rightarrow$ C and C_0 are isotopic.

By construction, the two possible framings on C_0 (even/odd) transform S^4 to $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$.

inter form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad$ $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \square$

Recall: Dehn surgery on M^3 (oriented 3-manifold) along $K \subset M^3$ and according to a framing $\varphi : \mathbb{T}^2 \xrightarrow[\text{diff.}]{\cong} \mathbb{T}^2$ is the manifold:

$$M(K,\varphi) := (M^3 \setminus \nu K) \cup_\varphi S^1 \times \mathbb{D}^2$$



$M^3 \setminus \nu K$ is a solid torus.

Note: In S^3 φ is given precisely by a pair of rel. prime integers. If K is oriented, define μ = positive meridian
λ = 0-framed longitude

(note: changing orientation of K changes orientations of both μ & λ so the orientation of K is irrelevant)

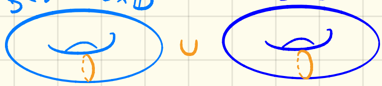
For $(p,q)=1$: $p\mu + q\lambda$ is a unique simple closed curve on $\mathbb{T}^2 = \partial(S^3 \setminus \nu K)$

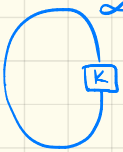
examples.

(a) $0 = \frac{0}{1} \rightsquigarrow 0 \cdot \mu + 1 \cdot \lambda$

U = unknot

This represents $S^3 \setminus \nu U \cong S^1 \times \mathbb{D}^2 \cup S^1 \times \mathbb{D}^2 = S^1 \times S^2$

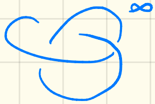


b) 

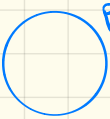
$\infty = \frac{1}{0}$

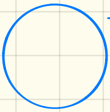
$= L(1,0) = S^3$

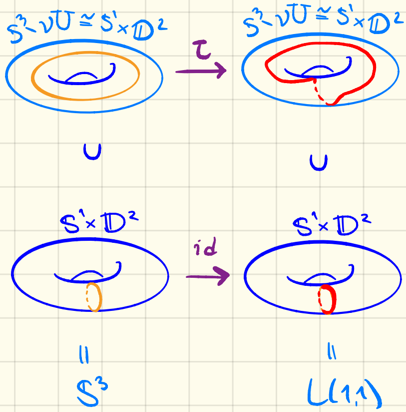
↖ any unknot

e.g.  $^{\infty} = S^3$

because: we remove a solid torus and glue it back same way.

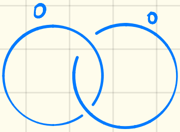
c)  $^{p/q}$ $= L(p,q)$ the lens space with $\pi_1 \cong \mathbb{Z}/p\mathbb{Z}$

d)  $^{\pm 1}$ $= L(\pm 1, 1) = S^3$ because



There is a map τ which is a diffeo, and agrees on boundary with gluing maps.
Namely:
τ = extension of Dehn twist along meridian to solid tori.

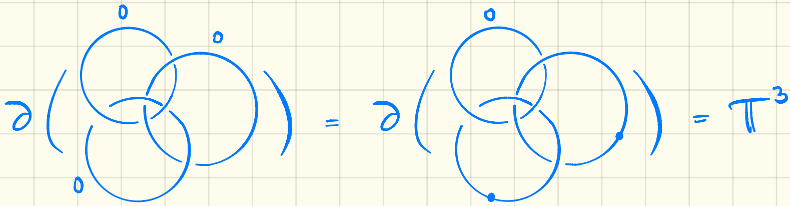
More generally: $L(p,q) \cong L(p, q + n\cdot p)$ for any n.
(try to prove similarly as $p = q = 1$)

(e)  $= S^3$

(Try to prove this without thinking about 4-mflds)

Big Insight: Integer-framed Dehn surgery is naturally the boundary of the 4-manifold $B^4 \cup \{2\text{-handles}\}$ and in that case Dehn surgery = surgery.

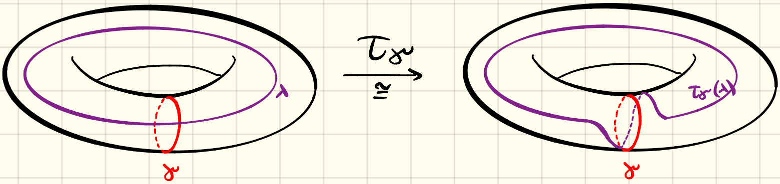
(f)



Theorem [Lickorish–Wallace '60's]

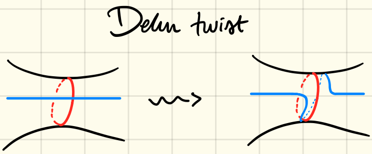
Every closed oriented 3-manifold is the result of Dehn surgery along some link in S^3.
The link can be chosen to have unknotted components and all the framings ± 1.

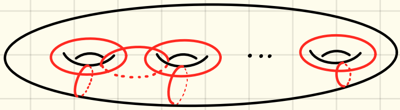
Defn. A Dehn twist τ_γ along $\gamma \subseteq \mathbb{T}^2$



Zicuorish twist theorem.

Σ_g closed orientable genus g surface
Any orient-pres. homeo of Σ_g is isotopic to some product of (positive or negative) Dehn twists about following $3g-1$ curves:





- we omit proof of this theorem.

Recall from last time:

THEOREM of Zicuorish - Wallace (1960's): Every closed orientable connected 3-mfld is the result of Dehn surgery along some link in S^3.

Lemma. Let H_g be genus g 3-dim^l handlebody. For any

$$f: \partial H_g \xrightarrow{\cong} \partial H_g$$

there exist pairwise disjoint $\{V_i\}_{i=1}^r$ and pairwise disjoint $\{V_i'\}_{i=1}^r$ solid tori in H_g s.t. f extends to a homeo

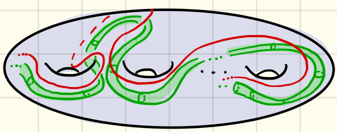
$$\bar{f}: H_g \setminus (\mathring{V}_1 \cup \ldots \cup \mathring{V}_r) \longrightarrow H_g \setminus (\mathring{V}_1' \cup \ldots \cup \mathring{V}_r')$$

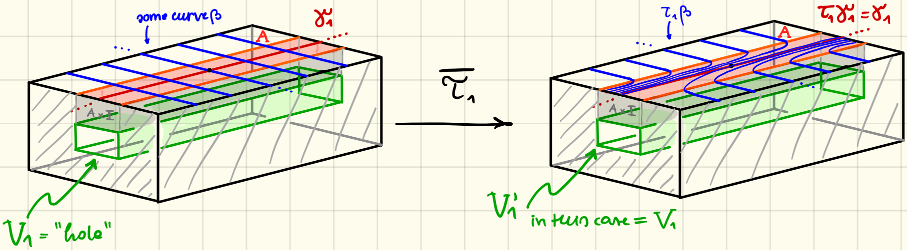
proof of Lemma. $f \simeq \prod_{i=1}^r \tau_i$ where τ_i are Dehn twist along γ_i

Assertion: suffices to prove for $\prod \tau_i$ (use the isotopy in a collar)

Now consider annulus neighborhood of γ_i. "Push" them into the handlebody.

example





Define $\overline{\tau}_1 := \begin{cases} \tau_1 \times id, & \text{on } A \times I \\ id, & \text{on } H_g \setminus V_1 \end{cases}$

Similarly define V_i for all i, and ensure V_i lies "below" V_{i-1}, so they are pairwise disjoint.

Define V_i' as follows: $V_r' = V_r$, $V_i' = \underbrace{\overline{\tau}_r \circ \overline{\tau}_{r-1} \circ \dots \circ \overline{\tau}_{i+1}}(V_i)$

this is needed since the curves γ_i may intersect with one another. ▨

Define $\overline{f} := \overline{\tau}_r \circ \overline{\tau}_{r-1} \circ \dots \circ \overline{\tau}_1$.

proof of THEOREM:

Let M^3 closed connected oriented.

$\exists g$ s.t. $M = H_g^1 \cup_f H_g^2$ Heegaard decomposition.

with $f: \partial H_g^1 \to \partial H_g^2$

Let $S^3 = H_g^1 \cup_{f'} H_g^2$ of same genus.

with $f': \partial H_g^1 \to \partial H_g^2$

$$\begin{array}{ccc} S^3 & & M^3 \\ \| & & \| \\ H_g^1 & \xrightarrow{id} & H_g^1 \\ \cup_{f'} & & \cup_f \\ H_g^2 & & H_g^2 \end{array}$$

induces the map $f \circ f'^{-1}: \partial H_g^2 \to \partial H_g^2$

By Lemma this extends away from some solid tori.

$\Rightarrow$ M is a Dehn surgery on some link in S^3

because:

it is obtained from $S^3 \setminus (V_1 \cup \dots \cup V_r)$ by gluing back $V_1' \cup \dots \cup V_r'$.

In fact: the link has unknotted components [but not the unlink!]

and framings are all ± 1. ← to see this need to see which curve in V_i is mapped to meridian of V_i'.

▨

Corollary. Any closed oriented connected 3-manifold is the boundary of an oriented simply-connected 4-manifold.

Question: Is there a minimal number of components for a link describing the given 3-manifold?

AUCKLY '97: two examples of $\mathbb{Z}HS^3$ which are not Dehn surgery on knot

HOM-KARAKURT-LIDMAN 2014: infinitely many examples of $\mathbb{Z}HS^3$ not surgery on a knot (all produced by surgery on a 2-comp link).

Open question: Is there a family of $\mathbb{Z}HS^3$'s which require arbitrarily many components in a surgery diagram?

Kirby's THEOREM. 1970's. Integer framed links in S^3 correspond to diffeomorphic 3-manifolds iff they are related by a sequence of:

ISOTOPY

HANDLE SLIDES

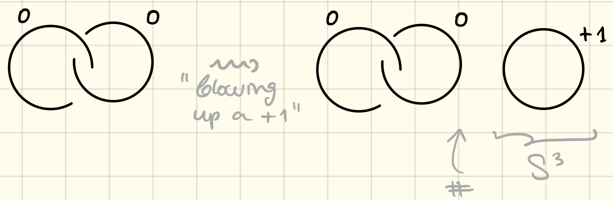
BLOW UP / DOWN.

Fenn-ROURKE improvement: handle slides are not necessary if general blow ups/downs allowed.

(simple) BLOW UP : add a ± 1-framed unknot split from your diagram

(simple) BLOW DOWN : remove —"—

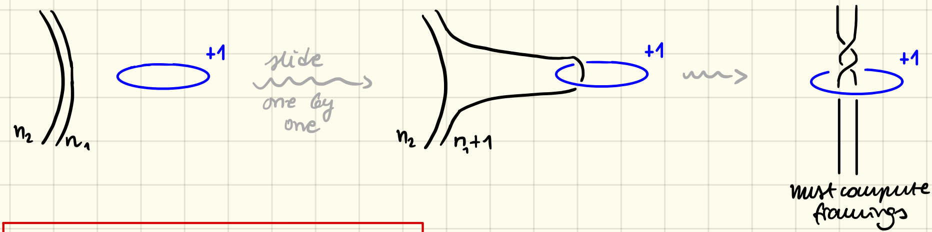
example.



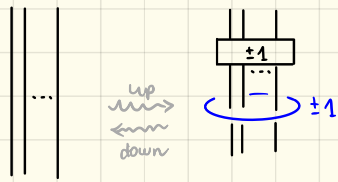
Recall. For 4-manifolds this is connecting sum with $\mathbb{C}P^2$ or $\overline{\mathbb{C}P}^2$ in the interior of the 4-manifold.

Now can slide over this O^{+1} :

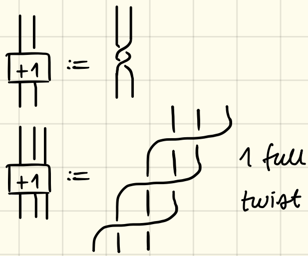
more strands:

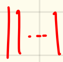


GENERAL BLOW UP / DOWN:



here box means:



note: if a handle h_i has k_i strands in the collection 
(algebraic count with signs), then blow up / down
changes its framing by $+k_i^2 \,/\, -k_i^2$ (check this using double-strand notation or inter. form)

proof of Kirby's theorem:

($\Rightarrow$) L_1, L_2 links

Let M_{L_1} and M_{L_2} be the corr. 4-manifolds with $\partial M_{L_1} \cong \partial M_{L_2}$.

Define

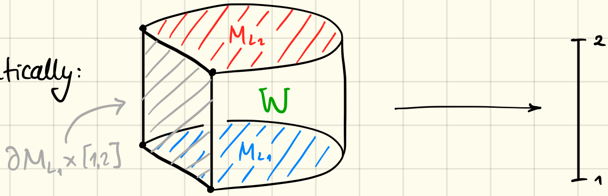
$$N^4 = M_{L_1} \cup (\partial M_{L_1} \times [0,1]) \cup -M_2$$

Now connect sum M_{L_1} with $\pm \mathbb{C}P^2$ until $\sigma(N) = 0$

signature of inter. form of N.

Thom's Theorem: $\sigma(N) = 0 \Rightarrow N^4 = \partial W^5$
connected oriented smooth 5-mfld

schematically:



Let $f : W \longrightarrow [1,2]$ be a Morse fn. s.t. $f^{-1}(i) = M_{L_i}$ $i = 1,2$
$f|_{\partial M_{L_1} \times [1,2]}$ projection.

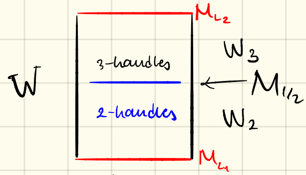
i.e.

W is built by attaching handles to M_{L_1}

Now we modify W without changing its boundary:

- W is connected $\Rightarrow$ cancel all 0-handles
  cancel all 5-handles
- Do surgery on circles to make W simply-connected
  (do surgery on generators of $\pi_1 W$ in interior of W)
  then do handle trading until No 1-handles.
  (i.e. replace 1-handles by 3-handles) (see Classes 26&27 for more details)
  Similarly, cancel all 4-handles.

$\Rightarrow$ Only 2- and 3-handles are left:



Let $M_{1/2}$:= the "middle level" of W i.e.

$$M_{1/2} = \partial(M_{L_1} \cup 2\text{-h})$$
$$= \partial(M_{L_2} \cup 3\text{-h})$$
upside down

All circles in M_{L_i} are null-homotopic.
By proposition from the last class:

$$M_{1/2} = M_{L_1} \# \mathop{\#}_{k_1} S^2 \times S^2 \# \mathop{\#}_{l_1} S^2 \tilde{\times} S^2$$
$$= M_{L_2} \# \mathop{\#}_{k_2} S^2 \times S^2 \# \mathop{\#}_{l_2} S^2 \tilde{\times} S^2$$

Now use the HWA2 (Bonus): $S^2 \times S^2 \# \mathbb{C}P^2 = \bigcirc^{+1} \bigcirc^{-1} \bigcirc^{+1}$

$$S^2 \tilde{\times} S^2 = \bigcirc_{+1} \bigcirc_{-1}$$

so we can go from L_1 to L_1'
L_2 to L_2' so that $M_{L_1'} \cong M_{L_2'}$.

Can I now go from handle decomposition L_1'
to handle decomposition L_2' > both are h.d. of $W_{1/2}$
using only handle slides?

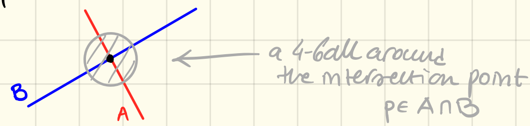
Kirby: using Cerf theory Yes.
(can avoid birth-death cancellation).

$\boxtimes$

Class 25 — Th. Jan 17

§ INTERLUDE: surfaces in 4-manifolds.

Given surfaces A, B in a 4-manifold



On the boundary S^3 of a small ball around p we find a link $A \cap S^3 \sqcup B \cap S^3$.

If this link were slice i.e. components bound pairwise disjoint smooth discs in $\mathbb{B}^4$ then we could modify A and B (preserving homology classes) by gluing on slice discs and removing bad point p.

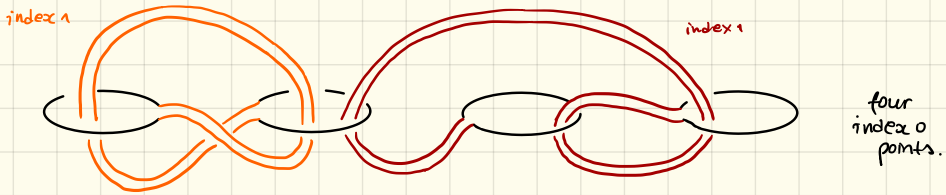
exercise. in the case of the above picture the link is NOT slice. Why?

Defn. A link $L \subseteq S^3$ is said to be ribbon if the components bound pairwise disjoint smooth discs in $\mathbb{B}^4$ perturbed so that they are Morse wrt radius function on $\mathbb{B}^4$ and have NO LOCAL MAXIMA.

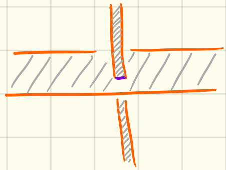
(recall from Lecture 8) and HW5

i.e. the discs have only index 0 and 1 critical points

Equivalently: "bottom up" a ribbon link is produced from an unlink fused together by some bands, which always reduce the number of components (otw would create genus).

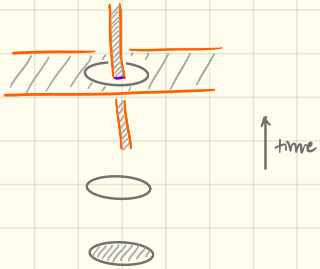


Equivalently : a ribbon link bounds a collection of ribbon discs in S^3
(recall HW5) i.e. a collection of immersed discs in S^3
whose only singularities are of the form



this is a ribbon intersection in S^3

Note:
can push a part of disc into B^4



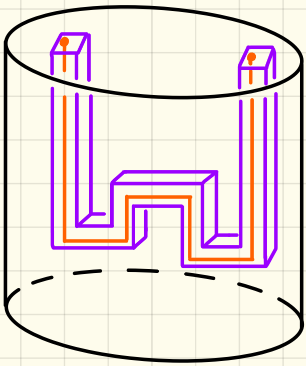
<u>Slice - ribbon Conjecture</u> (open!)
Any slice link is ribbon.

§ Kirby diagrams for ribbon disc complements.

A ribbon disc has a natural handle decomposition
n 0-handles and $(n-1)$ 1-handles for some n.

IN GENERAL: Given $(Y^m, \partial Y^m) \hookrightarrow (B^n, \partial B^n)$
then every k-handle of Y gives
a $(k+n-m-1)$-handle of $B^n \setminus \nu Y^m$
... open reg. nbhd.

for us $m=2, n=4$.
But let's look at $m=1, n=3$:

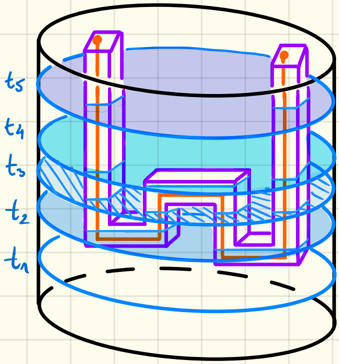


Let us determine the complement of the tubular neighbourhood (purple) of the orange curve.

– as a series of level pictures.

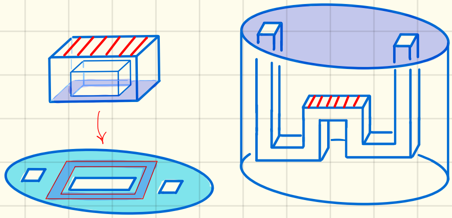
0-handles $\longleftrightarrow$ $(0+3-1-1) = $ 1-handles of complement

1-handles $\longleftrightarrow$ $(1+3-1-1) = $ 2-handes of complement



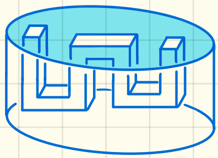
$t = t_5$

attach a 2-handle as "a roof"



$t = t_4$

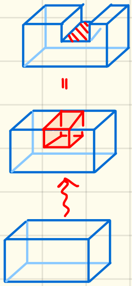
just make a depression (place your finger between two inner square-holes)

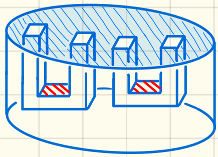


$t = t_3$

We attached two 1-handles to the complement

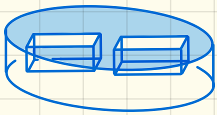
namely:



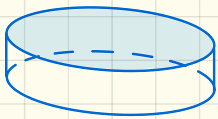


$t = t_2$

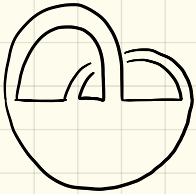
this is still a $\mathbb{B}^3$ with two "depressions"



$t = t_1$



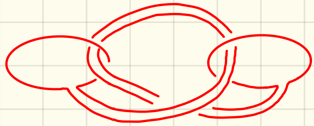
Complement of a surface $F \subseteq \mathbb{B}^4$



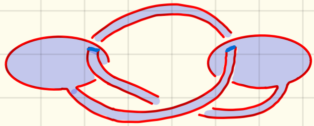
framings always 0 (need to examine above construction)

Ribbon disc complement:

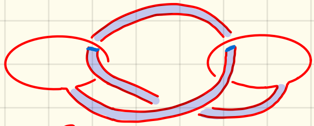
Stevedores knot



a choice of a ribbon disc for it

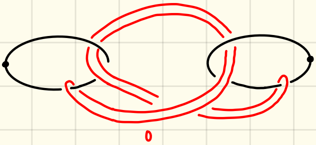


in S^3



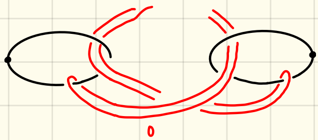
1-handle in $\mathbb{B}^4$ (move)

the complement in $\mathbb{B}^4$



! Same knot can have non-isotopic ribbon discs

another example for Stev. knot:



§ PREVIEW of next Lectures:

Wall '60's: If M_0 and M_1 are smooth simply-connected closed 4-mflds with isomorphic inter. forms then they are h-cobordant.

Freedman: 5-dim'l h-cobordisms are topologically a product.

Curtis-Hsiang-Freedman-Stong - Matveyev - Bižaca - Kirby

Class 26.

Jan 22 Tue

Open problem session: next Tuesday (Jan 29)

Theorem [Wall '60s]
If M_0, M_1 are smooth simply-connected closed 4-manifolds with isomorphic intersection forms, then they are (smoothly) h-cobordant.

Recall: h-cobordant means $\exists W^5$ smooth s.t. $\partial W = -M_0 \sqcup M_1$
and $i_i : M_i \hookrightarrow W$ is a homotopy equivalence. $i=0,1$

∇ not true for topological 4-manifolds:
e.g. $\exists$ a topological simply-conn. closed 4-mfld called $*\mathbb{CP}(2)$
it has inter. form $\langle +1 \rangle$, but it is not homeo to $\mathbb{CP}(2)$

(actually: $*\mathbb{CP}(2)$ and $\mathbb{CP}(2)$ are not even topologically cobordant (since they have different Kirby-Siebenmann invariants)
so already the first part of the proof fails)

Theorem [Freedman '80s]
Any smooth simply-connected h-cobordism is homeomorphic to a product.

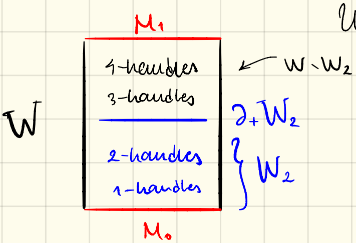
Corollary. Smooth simply-connected closed 4-manifolds with isomorphic inter. forms are homeomorphic.

proof of Wall's theorem:
$\sigma(M_0) = \sigma(M_1) \Rightarrow \sigma(-M_0 \sqcup M_1) = 0$
$\Rightarrow \exists W$ a cobordism from M_0 to M_1
$\left(\text{Thm: } \Omega_4^{or} \xrightarrow[\cong]{\sigma} \mathbb{Z}\right)$

Goal: improve W to a h-cobordism.

1) do surgery on circles in W to make $\pi_1 W$ trivial.
2) assume there are no 0- and 5-handles
3) assume there are no 1- and 4-handles: HANDLE TRADING.

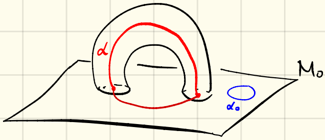
§ What is handle trading?



We trade every 1-handle for a 3-handle:

- note that M_i and W are simply conn.
- Let h be any 1-h. in W.

let α = (core of h) $\cup$ (an arc joining the feet of h)
Push α into $\partial_+ W_2$ (by transversality).

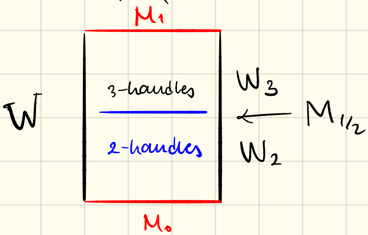


- Let α_0 be an unknot in $\partial_+ W_2$ away from all 1-, 2-h.
- Introduce to W a cancelling 2-/3-handle pair, where 2-handle is attached to α_0 with triv. framing.

- Note: α and α_0 are isotopic in $\partial_+ W_2$ since:
  - $\partial_+ W_2$ is simply-connected (because $W \setminus W_2$ & M_1 are simply-conn).
  - homotopy implies isotopy for loops in in 4-mfld.
- Use the 2-h. att. to α_0 to cancel h. (Remains a 3-handle.)

//

Back to proof.



Since M_0 and M_1 are simply-conn.
by our previous PROPOSITION: (see Class 24) (Jan 10)

$$M_{1/2} \cong M_0 \# k_0 S^2 \times S^2 \# \ell_0 S^2 \tilde{\times} S^2$$
$$\cong M_1 \# k_1 S^2 \times S^2 \# \ell_1 S^2 \tilde{\times} S^2$$

<u>Claim</u>. We can assume there are <u>no</u> $S^2 \tilde{\times} S^2$ summands, i.e. $\ell_0 = \ell_1 = 0$.

Corollary of the Claim (and proof so far). If M_0 and M_1 are smooth simply-conn closed with isomorphic inter. forms, then there is $k \in \mathbb{N}$ s.t.

$$M_0 \# k S^2 \times S^2 \underset{\text{diffeo}}{\cong} M_1 \# k S^2 \times S^2$$

Defn. We say that M_0 and M_1 are *stably diffeomorphic*.

Open Question: Is $k=1$ enough? all examples we know: yes.

proof of the claim. The inter. form Q_{M_i} is either even ($\forall x \in H_2 \;\; Q(x,x)$ even) or odd (not even)

A spin structure on a smooth manifold is a (homotopy class of a) trivialisation of the tangent bundle over the 1-skeleton that extends over the 2-skeleton.

Note: M is spin $\overset{\text{def}}{\iff}$ M has a spin structure $\iff$ $\underbrace{w_1(M)=0}_{M \text{ orientable.}}$ and $w_2(M)=0$

FACT: M^4 simply-conn. compact (not nec. closed). Then M^4 is spin iff Q_M is even.

FACT: The boundary of a spin manifold is spin.

Rokhlin Theorem: Any spin smooth 4-manifold with zero signature bounds a spin smooth 5-manifold.

Returning to proof: suppose Q_{M_0} even.

Then $-M_0 \sqcup M_1$ spin and has zero signature.
Use Rokhlin's theorem instead of Thom's to get spin cob. W
do surgery on circles with "correct" framing so W stays spin.

$\Rightarrow$ W_2 spin $\Rightarrow$ $M_{1/2}$ spin $\Rightarrow$ $Q_{M_{1/2}}$ even.

It follows that $Q_{M_{1/2}}$ cannot contain $Q_{S^2 \tilde{\times} S^2} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.
Hence:

$M_{1/2}$ has no $S^2 \tilde{\times} S^2$-summands.

Now suppose Q_{M_0} odd?

Proposition. If Q_M odd, then: $M \# S^2 \times S^2$ is diffeomorphic to $M \# S^2 \tilde{\times} S^2$

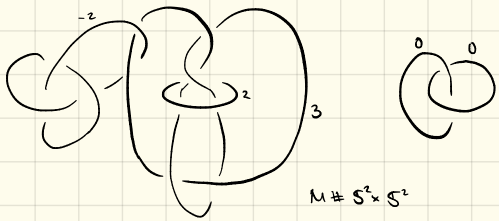
(Recall: we saw this for $M = \mathbb{C}P^2$)

proof sketch. Consider the case of no 1-handles in M.

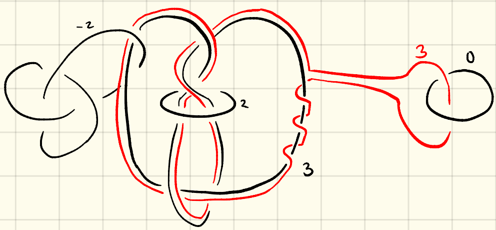
note: there must be at least one odd framing.

e.g.

$M :=$

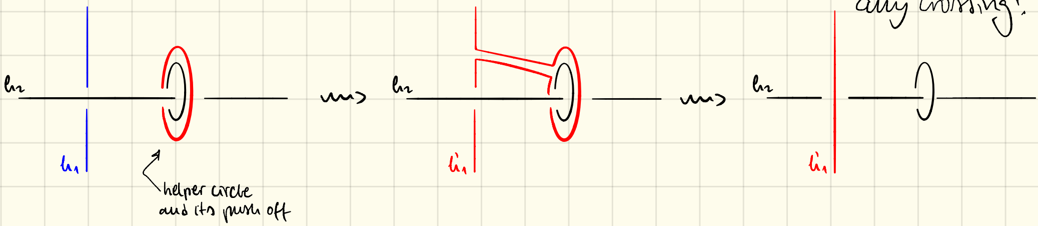


$M \# S^2 \times S^2$

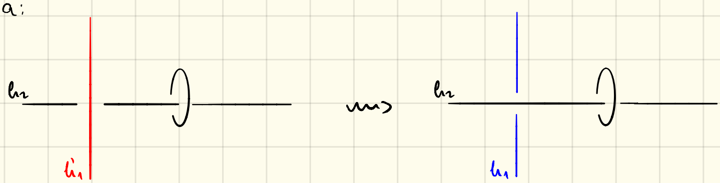


$\star$

Now: 0-framed meridians are called helper circles. They can change any crossing!

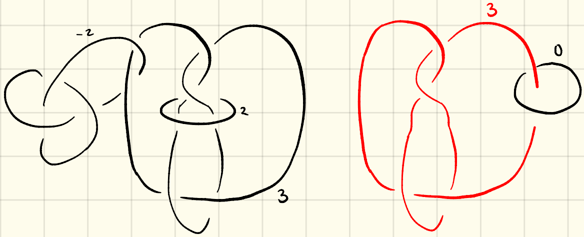


and vice versa:



The framing only changes when $h_2 = h_1$

Now can transform ★ to:



and undo all crossings in red circle until it is unknot.

Do handle slides over helper circle until it has framing +1. □

Class 27 Jan 24 THU

Theorem 1 [Wall] M_0, M_1 smooth closed simply-conn. $Q_{M_0} \cong Q_{M_1}$
Then $\exists k \geq 0$ s.t. $M_0 \# k S^2 \times S^2$ diffeo to $M_1 \# k S^2 \times S^2$

Theorem 2 [Wall] $M_0\ M_1$ smooth closed simply-conn. $Q_{M_0} \cong Q_{M_1}$
Then M_0 and M_1 are smoothly h-cobordant.

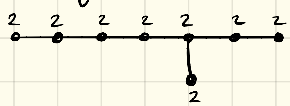
Theorem 3 [Wall] M smooth closed simply-conn
Q_M indefinite.
Then any automorphism of $Q_{M \# S^2 \times S^2}$
is realised by a self-diffeomorphism of $M \# S^2 \times S^2$.

Definition. Q_M is positive definite if $Q_M(\alpha, \alpha) > 0 \quad \forall \alpha \in H_2(M; \mathbb{Z})$
negative definite if $Q_M(\alpha, \alpha) < 0 \quad \forall \alpha \in H_2(M; \mathbb{Z})$
indefinite otherwise.

e.g. $[+1]$ pos.def, $[-1]$ neg.def, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ indefinite.
Standard definite forms are $\overset{n}{\oplus}[+1]$, $\overset{n}{\oplus}[-1]$ for $n \geq 1$.
E_8 is pos. def. but not standard:

$$\begin{bmatrix} 2 & 1 & & & \\ 1 & 2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 2 \end{bmatrix}$$

Dynkin diagram:



Aside: **Donaldson's Theorem:**
If a smooth closed simply-conn. 4-manifold
has definite intersection form,
then it must be one of standard definite forms.

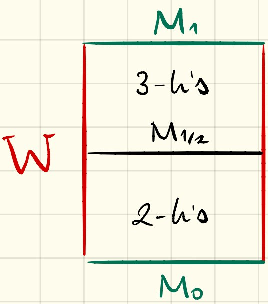
Remark. Later work shows that no need to restrict to simply-conn.

Note: In contrast, any symmetric unimodular integral bilinear form is realised as the intersection form of a closed simply-conn. topological 4-manifold. [Freedman]

proof of Thm 3 (idea): Wall identified the group of auto's of
$$Q_{M \# S^2\times S^2} = Q_M \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
and then realized them by self-diffeo's of $M \# S^2 \times S^2$. $\square$

proof of Thms 1 & 2 (cont^d):

We had built W = smooth cobordism between M_0 & M_1 with only 2- and 3-handles.



We observed: $M_{1/2} \cong M_0 \# k S^2 \times S^2$
$\cong M_1 \# k S^2 \times S^2$

(note: we proved odd case in case no 1-handles. (see last class) but in case there are 1-handles in M_0 there is another argument.)

$\square$ of Thm 1.

Plan for Thm 2:

Cut W along $M_{1/2}$ and reglue to get W' which is h-cobordism.

Note: It suffices to arrange that $\pi_1 W' = 1$ and $H_*(W, M_0)$ trivial.
(Whitehead - Hurewicz implies $M_0 \hookrightarrow W'$ htpy equiv.
Poincaré - Lefschetz implies $M_1 \hookrightarrow W'$ also htpy equiv)

$Q_{M_{1/2}}$ is indefinite as long $k \geq 1$ (otw we are done).

By Theorem 3 any automorphism of $Q_{M_{1/2}}$ is realized by a self-diffeo as long as ($k \geq 2$) or (Q_{M_0} is indefinite and $k \geq 1$)

Can accomplish this by adding a cancelling 2-/3-h pair.

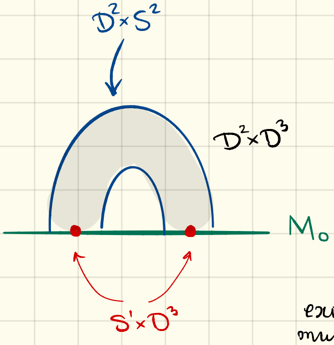
=> algebra is controlling geometry!

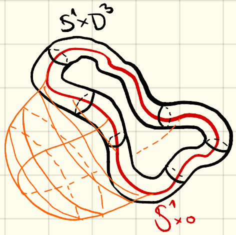
We choose the right automorphism of $Q_{M_{1/2}}$.

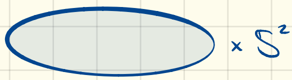
$$H_2(M_{1/2}; \mathbb{Z}) = H_2(M_0) \oplus \mathbb{Z}\langle \alpha_1, \bar{\alpha}_1, \dots, \alpha_k, \bar{\alpha}_k \rangle$$

core of 2-h

belt-sph of 2-h







glue $\partial \times S^2$ to $S^1 \times S^2 = \partial(S^1 \times D^3)$

exists since M_0 simply-conn.

this is $S^2 \times S^2 \setminus$ 4-ball.

$\Rightarrow$ We connected-sum M_0 and $S^2 \times S^2$

However, looking upside-down:

$$H_2(M_{1/2}; \mathbb{Z}) = H_2(M_1) \oplus \mathbb{Z}\langle \beta_1, \bar{\beta}_1, \dots, \beta_k, \bar{\beta}_k \rangle$$

belt sphere of upside-down 3-h = att. sphere of 3h

core of upside down 3-h

By thm. hypothesis $\exists \varphi : H_2(M_1) \xrightarrow{\cong} H_2(M_0)$ inducing isomorphism $Q_{M_1} \cong Q_{M_0}$

Extend it by sending

$$\beta_i \xmapsto{\bar\varphi} \alpha_i \qquad \forall i$$

e.g. $\bar\varphi(\bar\beta_i) = \bar\alpha_i$

because $Q(\underbrace{\bar\varphi(\beta_i)}_{=\alpha_i}, \bar\varphi(\bar\beta_i)) = Q(\beta_i, \bar\beta_i) = 1$

Then by construction: "β_i intersects $\bar\alpha_i$ once and $\bar\alpha_j$ zero times if $i \neq j$"

Thm 3 of Wall: $\tilde\varphi : M_{1/2} \longrightarrow M_{1/2}$ realizing $\bar\varphi$.

Now build $W' := W_2 \cup_{\tilde\varphi} W_3$

Get that att. sph (3-h) intersects belt-sphere (2-h) alg. once,
and all other belt spheres alg. zero times. ← for a single 2-h.

(Warning: we are changing W to a completely different cobordism W')

$\square$

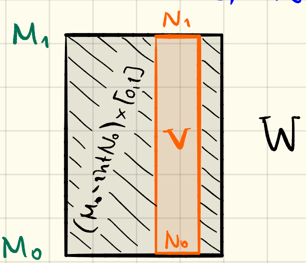
[Curtis-Hsiang, Freedman-Stong, Matveyev-Kirby-Bižaca]

M_0, M_1 smooth closed simply-conn, smoothly h-cobordant via W

Then there exists a sub h-cobordism $V \subset W$

between submanifolds $N_i \subset M_i$ s.t.

1) N_i, V are compact and contractible
2) $W \setminus \operatorname{int} V$ is diffeo to $(M_0 \setminus \operatorname{int} N_0) \times [0,1]$
3) N_0 and N_1 are diffeo via a diffeo which is an involution on the boundary ∂N_0.



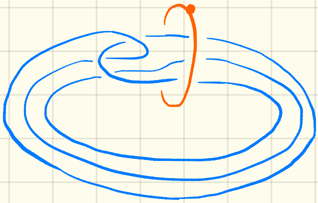
Definition. A cork is a compact smooth contractible 4-mfld A with a diffeo

$$f : \partial A \longrightarrow \partial A$$

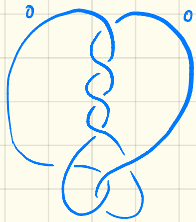
which does not extend to a self-diffeo of A.

note: some require f to be involution
some require A to be "Stein"

example.



has $\partial =$



Akbulut cork (see HW A3.1)

Cork Thm. Any two homeomorphic smooth closed simply-conn. 4 mflds differ by a cork twist,

i.e. remove a cork from one
reglue via an involution on the ∂.

proof of Cork Thm. Apply Wall's Thm 2, then CHFSMKB say:

$$M_1 = (M_1 \setminus \operatorname{int} N_1) \cup N_1$$

$$\cong$$

$$(M_0 \setminus \operatorname{int} N_0) \cup N_1 \quad \swarrow \text{cork}$$

$$\cong$$

$$N_0$$

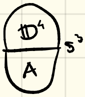
$\square$

Open Problems Session

Jan 29 TUE

1) Poincaré Conjecture: Any smooth 4-manifold Σ^4 which is a homotopy 4-sphere is diffeo. to S^4.

$\Leftrightarrow$ $\quad \bar{A} \simeq *,\ \partial \bar{A} = S^3 \Rightarrow A \cong D^4$

proof of $\Rightarrow$:  $\left\{ \begin{array}{l} \text{smooth} \\ \overset{PC}{\Longrightarrow} S^4 \end{array} \right.$.

<u>Palais</u>: any two maps of D^n into a smooth n-mflds are isotopic up to reflection.

Hence $A \cong D^4$

proof of $\Leftarrow$: $\Sigma \setminus \mathring{D}^4_{\text{small}} = \bar{A}$, then $\bar{A} \simeq *$ so $\bar{A} \underset{f}{\cong} D^4$

Then $\Sigma^4 = (\Sigma^4 \setminus D^4) \cup_{S^3} D^4 \cong D^4 \cup_{fl} D^4$ $\quad fl_{S^3}: S^3 \circlearrowleft$

<u>Cerf</u>: $\mathrm{Diff}^+(S^3) \xleftarrow{\simeq} SO(4)$ (special case of π_0 of Laudenbach–Poenaru)

Hence $\Sigma^4 \cong S^4$.

2) Schoenflies Conjecture: A smooth $S^3 \hookrightarrow S^4$ bounds a smooth $D^4 \subseteq S^4$.

<u>note</u>: $i: S^3 \hookrightarrow S^4$ has normal bundle $S^3 \times I \hookrightarrow S^4$

Have by duality $S^4 \setminus i(S^3) = A \sqcup B$.

Put back boundary

$\bar{A}, \bar{B} \subseteq S^4$ are smooth 4-mflds with $\partial \cong S^3$

$\bar{A} \simeq A \simeq *$ because they are homology-D^4 (MV) and π_1 trivial because can use Seifert-Van-Kampen (have collar to get open)

Hence: Poincaré $\Rightarrow$ Schoenflies.

* Θ_n = smooth oriented homotopy n-sphere / diffeo.

\# gives comm. monoid with unit.

For all $n \geq 5$ this is a finite group! since $S^n \underset{h\text{-cob}}{\sim} \Sigma \# -\Sigma$

group of htpy spheres

For $n=4$:

$\Theta_4^{inv} \subseteq \Theta_4$ consists precisely of those Σ^4 s.t.

$$\Sigma \setminus \mathring{D}^4 \subseteq S^4.$$

Poincare conj: $\Theta_4 = \{S^4\}$
Shrenfuss conj: $\Theta_4^{inv} = \{S^4\}$.

Is group completion trivial?
i.e. $\forall \Sigma \in \Theta_4$ is there $\Sigma' \in \Theta_4$ s.t. $\Sigma \# \Sigma' = \Sigma'$.

<u>Aside.</u> π_n^{st} = n-th homotopy group of spheres.

Pontrjagin-Thom: $\pi_n^{st} \cong \Omega_n^{fr}$ bordism gp of framed n-mflds

Have a map: $\Theta_n \xrightarrow{\alpha} \Omega_n^{fr} /$ change of framing

i.e. homotopy spheres can be framed

$\ker\alpha$ and $\operatorname{coker}\alpha$ are well-understood finite ab. gps (cf Kervaire invt. problem)

Cor. Θ_n is a finite group.

* closed 4-manifolds with ∞-ly many smooth structures: e.g. $2\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$

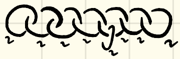
There is no known 4-manifold with finitely many sm. str.

We don't know for e.g. S^4, $\mathbb{CP}^2$, $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, $S^2 \times S^2$

$\mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \cong S^2 \tilde{\times} S^2$

3) Is the E_8-4-manifold a CW complex?

recall: closed top. simply-conn 4-mfld that is not smoothable

i.e.  $\cup_\Sigma \; C_\Sigma$ ← Freedman's 4-ball ($\simeq *$ but only topological)

Poincare homology sphere

note: E_8-plumbing in dim 8 (so using TS^4) is compact W^8 with boundary an exotic 7-sphere.
This is generator of $\Theta_7 = \mathbb{Z}/28\mathbb{Z}$

Note: there is no handle structure on E_8-4-manifold (Kirby–Siebenmann invt) and no triangulation (Casson invariant)

Note: any closed d-mfld is homotopy equivalent to a finite d-dim'l CW complex.

Is C_Σ CW complex? Is $*\mathbb{CP}^2$ CW complex?

← $\simeq \mathbb{CP}^2$ but $KS \neq 0$.

4) Is any closed 4-mfld M homeomorphic to $M_{smooth} \cup_\Sigma C_\Sigma$?
= can you cut every top mfld along a homology sphere into a smooth 4-mfld and a contractible piece.

5) 11/8-conjecture: $\dfrac{b_2 M}{|\sigma M|} \geq 11/8$ for M^4 closed smooth indefinite λ_M (even).

Recall: int. forms < definite: Donaldson; indefinite: classified by rank, σ, parity < even $kE_8 \oplus l\mathcal{H}$; odd $k[1] \oplus l[-1]$

We can realize $k[1] \oplus \ell[-1]$ by $\#^{k} \mathbb{C}P^2 \#^{\ell} \overline{\mathbb{C}P^2}$

Alg. geom $\Rightarrow$

$$2E_8 \oplus 3\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightsquigarrow \text{ has ratio } \frac{b_2}{|\sigma|} = \frac{11}{8}$$

Can get $\geq \frac{11}{8}$. Can we get sth between $\frac{10}{8}$ and $\frac{11}{8}$?

Furuta: can't get smaller.

Recall: $\lambda_2 : \pi_2 M \times \pi_2 M \longrightarrow \mathbb{Z}[\pi_1 M]$ inter. "number"

$$\lambda_3 : \left\{ \begin{array}{l} \text{three spheres} \\ \text{with pairwise} \\ \lambda_2 = 0 \end{array} \right\} \longrightarrow \frac{\mathbb{Z}[\pi_1 M \times \pi_1 M]}{\text{relations}}$$

$$\cap$$

$$\pi_2 M \times \pi_2 M \times \pi_2 M$$

Q: If M is closed $^{\text{smooth}}$ does λ_2 determine λ_3?

Yes if $\pi_1 M = 0$.

(note: λ_3 for topol. 4-mflds determines KS

Thm. M^4 closed topol, $\pi_1 M = 0$, odd λ_M

$\exists c \in H_2 M$ s.t.

$$\lambda(x,x) \equiv \lambda(c,x) \pmod 2 \ \forall x.$$

Then:

$$KS(M) = \frac{\lambda(c,c) - \sigma(M)}{8} + \tau(c)$$

immersed sphere

λ_M even: $KS(M) = \frac{\sigma(M)}{8} \pmod 2$

(o is char)

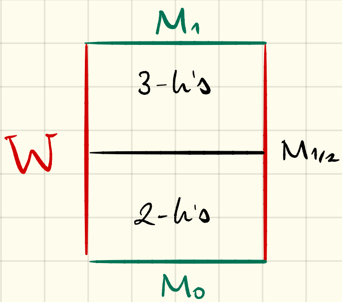
Class 29 Jan 31 THU

§ Overview of M. Freedman's work on topological 4-mflds.

§ h-cobordism

Thm. [Casson, Freedman.] (first part by Casson) Let W be a smooth h-cobordism between simply-connected 4-manifolds M_0 and M_1. Then W is homeomorphic to $M_0 \times [0,1]$.

proof. By standard arguments, assume W has only 2- and 3-handles.



$H_*(W, M_0) = 0 \Rightarrow$ By h-slides assume 2-/3-handles "cancel algebraically".

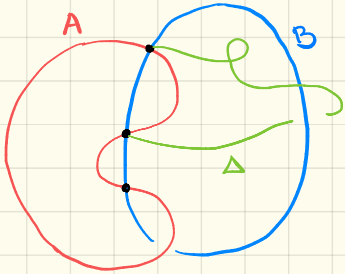
Recall: in high dim's "geometry follows from algebra".
Here: we don't have the Whitney trick.

Let $\{A_i\}$ be attaching spheres for 3-handles
Let $\{B_i\}$ be belt spheres for 2-handles. $A_i, B_i \subseteq M_{1/2}$

algebraic cancellation means: $\lambda(A_i, B_j) = \delta_{ij}. \quad \forall i,j$

Plan: realise λ geometrically by approximating the Whitney trick.

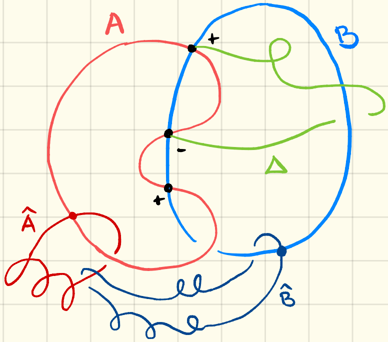
For convenience, suppose there is only a single pair A, B.



in $M_{1/2}$ which is simply-conn.
Every loop null-htpic
Δ = trace of null-htpy
- can make it immersed
- can assume framed
(by doing boundary twists if necessary.)

$$\text{Recall: } \quad M_{1/2} \cong M_0 \# \underbrace{S^2}_{\hat{B}} \times \underbrace{S^2}_{B}$$
$$\cong M_1 \# \underbrace{S^2}_{\hat{A}} \times \underbrace{S^2}_{A}$$

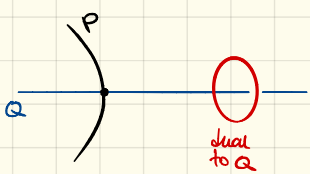
Problem 1: Get $\operatorname{int}\Delta \subseteq M_{1/2} \setminus (A \cup B)$.



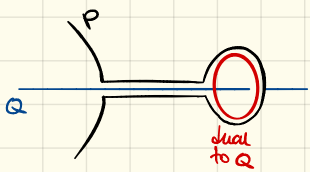
Want to arrange this by tubing into geometrically dual spheres $\hat{A}, \hat{B}$.

i.e.
$$\left.\begin{aligned} &\hat{A} \pitchfork A = \text{single pt} \\ &\hat{A} \pitchfork B = \emptyset \\ &\hat{B} \pitchfork B = \text{single pt} \\ &\hat{B} \pitchfork A = \emptyset \\ &\hat{A}, \hat{B} \text{ framed immersed spheres} \end{aligned}\right\} *$$

TUBING:



tube $\rightsquigarrow$ into dual

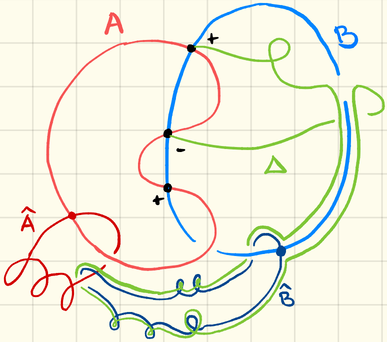


Note: As a consequence:

$$\pi_1(M_{1/2} \setminus (A \cup B)) = 1$$

This is called:

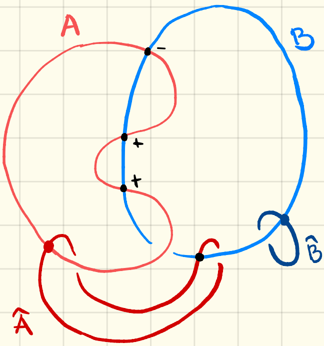
$A \cup B$ is "π_1-negligible".



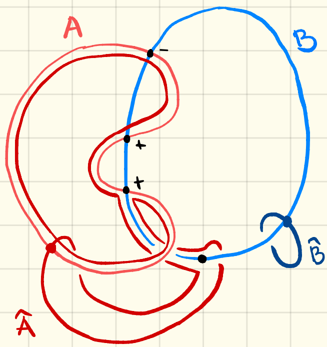
after we tube.

So problem is to find geometrically dual spheres.
so
find some $\hat{A}, \hat{B}$ sat. properties $*$.
can change A, B in the process. [but only up to isotopy!]

Problem. Given $\hat{A}$ might intersect B (symmetric: $\hat{B}$ can inter. A)



tube



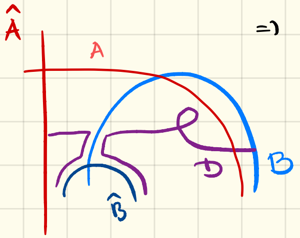
1) Tube $\hat{A}$ into A to make $\lambda(\hat{A}, B) = 0$
since:

$\lambda(A, B) = 1$ can pick one pt $+$, ← tube into that one.
all others cancelling pairs.

2) Now can pair inter. of $\hat{A}$ and B and find Whitney discs D
want: remove inter. of D with A and B.

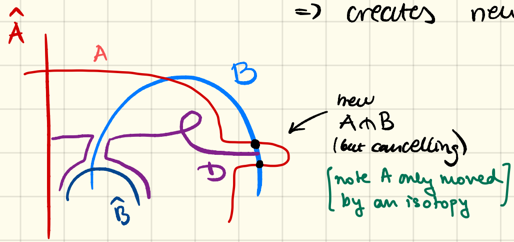
- remove inter. with B by tubing D into $\hat{B}$.

$\Rightarrow$ no inter with B but maybe new A inter.



- remove inter. with A by finger moves towards B

$\Rightarrow$ creates new cancelling A & B intersections.



3) Do a framed immersed Whitney move of $\hat{A}$ over D

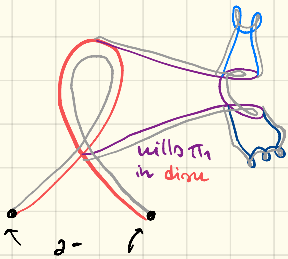
$\Rightarrow$ resulting $\hat{A}$ does not intersect B.

$\Longrightarrow$ We have extra inter. points of A and B paired by framed immersed Whitney discs Δ s.t.

$$\Delta \cap A = \Delta \cap B = \emptyset .$$

Wow wait Δ embedded!

Brilliant idea of Casson

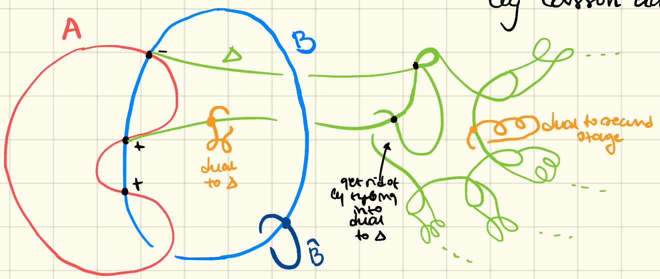


Casson tower of height 3.

Casson handle is a "Casson tower of infinite height" (4-dim'l!)
- it is simply-connected
- $\partial^- CH$ = solid torus.

Freedman: 1982. $(CH, \partial^- CH)$ is homeomorphic to $(D^2 \times \mathring{D}^2, S^1 \times D^2)$.

the rest of the proof: Pair intersections of A and B by Casson handles in $M_{1/2} \setminus A \cup B$



By Freedman: $\exists$ topologically embedded Whitney discs.

$\square$

Poincaré Conjecture: Any homotopy 4-mfld is homeomorphic to S^4.

proof. Let Σ be a smooth htpy S^4.

$\overset{\text{Wall}}{\Longrightarrow}$ Σ and S^4 are smoothly h-cobordant.

$\overset{\text{h-cob.}}{\Longrightarrow}$ Σ and S^4 are homeomorphic.

If Σ is a topological htpy S^4 (top. 4-mfld with htpy $= \pi_* S^4$)
then build a "proper" h-cobordism between Σ-pt and S^4-pt.

(proper h-cob: $\partial^{\pm} W \hookrightarrow W$ are proper htpy equiv.)

Or: alternatively use Quinn: 5-mflds have topol. handlebody structure.

$\Rightarrow$ category-preserving h-cobordism theorem.

(i.e. top. h-cob $\Rightarrow$ homeo^c to product)

so can build h-cobordism from Σ to S^4 using surgery theory.

Note: Freedman implies surgery theory works in dim 4 topologically. $\square$

other consequences: normal bundles
transverse intersections
connect-sum of top. 4-manifolds.

§ Exotic $\mathbb{R}^4$'s

$\mathbb{R}^n$ has a unique smooth structure $n \neq 4$.
and has uncountably many smooth structures if $n = 4$.

Defn. A smooth mfld $\mathcal{R}$ is said to be an exotic $\mathbb{R}^4$ if

$$\mathcal{R} \underset{\text{homeo}}{\approx} \mathbb{R}^4 \quad \text{but } \underline{\text{not}} \quad \mathcal{R} \underset{\text{diffeo}}{\cong} \mathbb{R}^4.$$

Defn. A knot $K \subset S^3$ is topologically slice if it bounds a flat disk

$$\Delta \subseteq B^4$$

i.e. $\forall p \in \Delta \quad (\nu_{p \subset B^4}, \nu_{p \subset D}) \approx (D^4, D^2)$

FACT: $\exists$ ∞ly many knots which are smoothly but not top. slice.

Example: any Alex. polyn. 1 knot is top. slice.

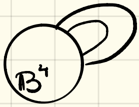
Wh.d (LHT) = 

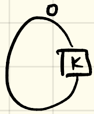
top. but not sm. slice.

§ Construction of an exotic $\mathbb{R}^4$.

Let K be a knot that is top. slice, but not sm. slice.

Let

$X_K :=$  $\leftarrow$ 2-handle attached along K with 0-fr.

Kirby diag 

FACT: $X_K \xrightarrow[\text{flat}]{\text{sm.}} \mathbb{R}^4_{std} \iff K$ is $\begin{matrix}\text{sm.}\\\text{top.}\end{matrix}$ slice.

K is top. slice $\implies X_K \xrightarrow{\text{flat}} \mathbb{R}^4$

We construct a sm. str on $\mathbb{R}^4$ as follows:

$\mathbb{R}^4 \setminus \operatorname{int}(X_K)$ is connected non-compact mfld.

$\xRightarrow{\text{Quinn}}$ it has a smooth structure.

We know X_K has its own smooth structure.

∂X_K and $\partial(\mathbb{R}^4 \setminus \operatorname{int}(X_K))$ are homeomorphic

Want to glue them together using a diffeomorphism.

need Thm of Bing-Moise: Any homeo of a 3-mfld is isotopic to a diffeo.

$\implies$ get a smooth str. on $\mathbb{R}^4$, call this $\mathcal{R}$.

Note: by construction $X_K \xrightarrow{\text{sm}} \mathcal{R}$, so $\mathcal{R} \underset{\text{diffeo}}{\not\cong} \mathbb{R}^4_{std}$ since K is not sm. slice.