

RICE UNIVERSITY

**Casson Towers and Filtrations of the Smooth Knot
Concordance Group**

by

Arunima Ray

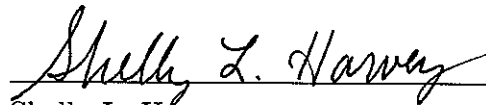
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ABSTRACT

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by

Arunima Ray

The 4-dimensional equivalence relation of *concordance* (smooth or topological) gives a group structure on the set of knots, under the connected-sum operation. The n -solvable filtration of the knot concordance group (denoted \mathcal{C}), due to Cochran–Orr–Teichner, has been instrumental in the study of knot concordance in recent years. Part of its significance is due to the fact that certain geometric attributes of a knot imply membership in various levels of the filtration. We show the counterpart of this fact for two new filtrations of \mathcal{C} due to Cochran–Harvey–Horn, the positive and negative filtrations. The positive and negative filtrations have definitions similar to that of the n -solvable filtration, but have the ability (unlike the n -solvable filtration) to distinguish between smooth and topological concordance. Our geometric counterparts for the positive and negative filtrations of \mathcal{C} are defined in terms of Casson towers, 4-dimensional objects which approximate disks in a precise manner. We establish several relationships between these new Casson tower filtrations and the various previously known filtrations of \mathcal{C} , such as the n -solvable, positive, negative, and *grope* filtrations. These relationships allow us to draw connections between some well-known open questions in the field.

To Jerry, Tim, Natalie, and Jeff, who keep me going.

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There are so many people to whom I am deeply indebted for guiding me and encouraging me on on the long road to this PhD.

I would not be here if not for my parents and my brother, who never let me think that anything was beyond my reach. I am grateful to my grandfather who, when I was a toddler, introduced me to ‘Egyptian mathematics’ – a fun game where I unknowingly solved addition and subtraction problems using circles. *Tabatai*, due largely to those happy hours spent with you as a child, I have always thought of math as fun; I wish you could be here with me today.

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Wendy Pogożelski, who mentored me and introduced me to research, and Harold Hoops, who teaches the perfect problem-solving based cell biology course—you made it hard to leave biochemistry! Many thanks are also due to my fellow math majors and Math Learning Center tutors, particularly Aaron Young, Cary Riina, and John Reynolds, who were a pleasure to work with and talk math with. Outside of mathematics, Geneseo was also the place where I made some wonderful friends who played an integral role in my success; Shannon, Shiv, Muting, Carla, Onur, my fellow resident assistants, particularly Cindy Malley, Colleen Bennett, and Alicia Housel, I miss all of you.

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One often hears of the impersonal and harsh nature of graduate school; for me, this is purely theoretical since Rice and the Rice math department have been universally good to me. The warm and supportive character of the Rice math department has been apparent to me since I was a prospective graduate student. As a first and second year student I received tons of help from my senior students, particularly Christopher Davis and Robert Vance; Chris in particular has been almost a second advisor to me. My fellow ‘fifthies’ Natalie Durgin, Darren Ong, Paul Munger, and

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Introduction

1.1 Background

A *knot* is the image of a smooth, oriented embedding $S^1 \hookrightarrow S^3$. In practice, one can think of a knot as simply a closed loop in space, such as those pictured in Figure 1.1. Similarly, a *link* is the image of a smooth, oriented embedding of an ordered collection of S^1 's, i.e. each component of a link is a knot. Two knots (or links) are *isotopic* if we can smoothly deform one to the other in 3-space. Isotopy gives an equivalence relation on the set of knots, and from now on, we will often use the word ‘knot’ to refer to an isotopy class of knots. For example, the two left-most pictures in Figure 1.1 are of the same knot, called the *unknot* or the *trivial knot*.

Knots and links are closely related to 3-dimensional *manifolds*—spaces that lo-

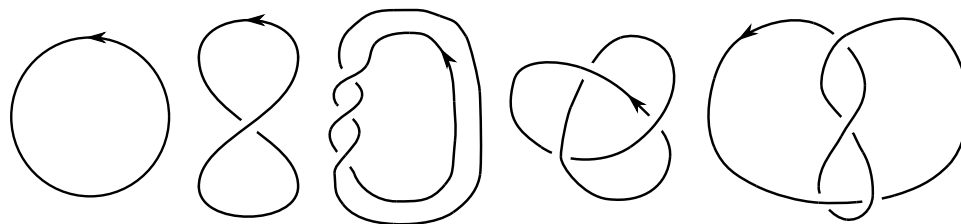


Figure 1.1: Examples of knots.

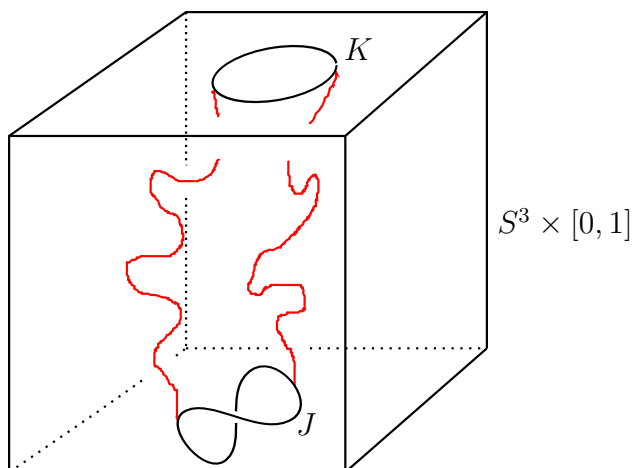


Figure 1.2: Concordance of knots.

cally resemble \mathbb{R}^3 —as we can see in the following famous theorem: any closed, connected, orientable 3-manifold can be obtained from the 3-sphere by performing a certain operation (*surgery*) on S^3 along some link [Lic62, Wal60]. Briefly, this involves cutting out a thickening of the knot from S^3 and then gluing the pieces back in a non-trivial manner. Manifolds (of various dimensions) form a very important topic of mathematical study and examples of manifolds abound in theory and practice alike.

1.1.1 Knot concordance

Two knots K and J are said to be *concordant* if they cobound a smooth annulus in $S^3 \times [0, 1]$ (here we consider K to be contained in $S^3 \times \{0\}$ and J in $S^3 \times \{1\}$). A schematic picture is shown in Figure 1.2. A knot is called *slice* if it is concordant to the unknot or equivalently if it bounds a smooth, properly embedded disk in B^4 . Such a disk bounded by a slice knot in B^4 is called a *slice disk*. Since a knot is trivial (i.e. isotopic to the unknot) exactly when it bounds an embedded disk in S^3 , being a slice knot is clearly a generalization of being a trivial knot. We can extend the notion of concordance and sliceness to links. Two links are concordant if there is a system of smooth annuli between them in $S^3 \times [0, 1]$ and a link is slice if its components bound

a collection of disjoint smooth disks in B^4 .

Much like how knots under the 3-dimensional equivalence relation of isotopy are related to 3-manifolds, knots under the 4-dimensional equivalence relation of concordance are closely related to 4-dimensional manifolds. This can be seen when one chooses to study a 4-manifold by understanding disjoint embeddings of 2-manifolds within it; this is a natural approach since, by analogy, the torus can be identified as being the only surface on which any two non-isotopic homologically non-trivial connected 1-manifolds must intersect. Two generic 2-manifolds within a 4-manifold intersect at isolated points. We would like to modify the 2-manifolds to make them disjoint. Since 4-manifolds are locally like \mathbb{R}^4 , we can find a small 4-dimensional ball V around each point of intersection. The 2-manifolds intersect the boundary of V (i.e. a 3-sphere) in a closed 1-manifold of potentially several components, namely, a link! Therefore, if the link were slice, we would be able to use the system of slice disks for it to obtain disjoint 2-manifolds.

There is a natural binary operation on the set of knots known as the *connected sum operation*, depicted in Figure 1.3. The set of knots, modulo concordance, under the connected sum operation forms an abelian group called the *knot concordance group*, denoted by \mathcal{C} . We will often use the same letter to denote a knot K and its concordance class.

There is a parallel theory of concordance in the topological category. In particular, two knots K and J are said to be *topologically concordant* if they cobound a

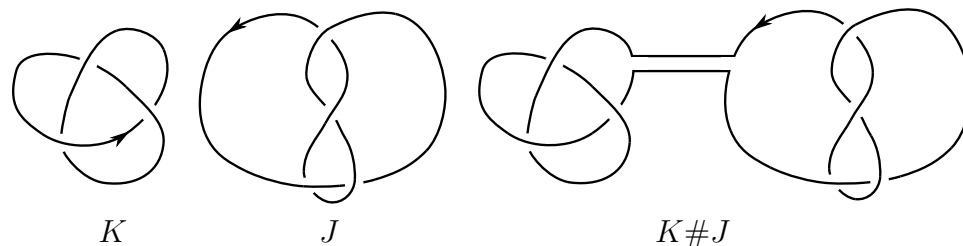


Figure 1.3: Connected sum of knots.

topologically embedded locally flat annulus in $S^3 \times [0, 1]$, and a knot is called *topologically slice* if it is topologically concordant to the unknot (or equivalently, if it bounds a proper, topologically embedded, locally flat disk in B^4). Clearly, any slice knot is topologically slice. There exist infinitely many knots which are topologically slice but not smoothly slice (see, for example, [Gom86]).

The marked dissimilarity between the smooth and topological categories is typical of dimension four. In some sense, one may consider dimension four as a boundary case between low and high dimensions: there are enough dimensions for the topology to exhibit complex behavior, but not enough space for our tools to work. This behavior is exemplified by the following: a closed manifold of dimension three or lower admits exactly one smooth structure; a closed manifold of dimension five or higher admits at most finitely many distinct smooth structures; however, a closed 4-manifold may have infinitely many distinct smooth structures [GS99, pp. 6–7].

The 3-dimensional study of knots frequently focuses on quantifying ‘how close a knot is to being unknotted’ through the use of various invariants such as Seifert genus, crossing number, unknotting number, etc. (see any introductory knot theory book, such as [Ada04] for more on these invariants). Similarly, the 4-dimensional study attempts to assess ‘how close a knot is to being slice’. In 2003, this notion was formalized when Cochran–Orr–Teichner [COT03] introduced the *n-solvable filtration* of \mathcal{C} :

$$\cdots \subseteq \mathcal{F}_{n+1} \subseteq \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}_0 \subseteq \mathcal{C}$$

and showed that the lower levels of the filtration encapsulate the information one can extract from various classical concordance invariants, such as algebraic concordance class, Levine–Tristram signatures, Casson–Gordon invariants, etc. That is, the deeper a knot is within the *n-solvable filtration*, the closer it is to being slice. Studying filtrations gives us a way of understanding the structure of \mathcal{C} , a large unwieldy object,

in terms of smaller (and hopefully simpler) pieces.

Part of the justification for the naturality of the n -solvable filtration is its close relationships with several more geometric filtrations of \mathcal{C} . In particular, certain geometric attributes of a knot imply membership in various levels of the n -solvable filtration, as seen in the following theorem.

Theorem 1 (Theorems 8.11 and 8.12 of [COT03]). *If a knot K bounds a grope of height $n + 2$, then K is n -solvable. If a knot K bounds a Whitney tower of height $n + 2$, then K is n -solvable.*

Recall that a knot K is slice if it bounds a disk in B^4 . We may therefore approximate sliceness in either of the following ways:

1. by considering objects bounded by K in B^4 which approximate disks
2. by considering disks bounded by K in manifolds that approximate B^4

The definition of the n -solvable filtration follows the second paradigm i.e. we filter knots based on certain properties of 4-manifolds within which they bound disks, while gropes and Whitney towers are approximations of a disk in a precise manner i.e. the taller a grope or Whitney tower bounded by a knot, the closer the knot is to being slice (see Chapter 2 for precise definitions).

While the n -solvable filtration has been an invaluable tool in the study of \mathcal{C} , it fails to distinguish between smooth concordance classes of topologically slice knots, i.e. if K is topologically slice, K is n -solvable for all n . To address this shortcoming, Cochran–Harvey–Horn [CHH13] recently introduced a new pair of filtrations of \mathcal{C} , the *positive* and *negative* filtrations:

$$\cdots \subseteq \mathcal{P}_{n+1} \subseteq \mathcal{P}_n \subseteq \cdots \subseteq \mathcal{P}_0 \subseteq \mathcal{C}$$

$$\cdots \subseteq \mathcal{N}_{n+1} \subseteq \mathcal{N}_n \subseteq \cdots \subseteq \mathcal{N}_0 \subseteq \mathcal{C}$$

(see Chapter 2 for precise definitions) which have been successful at distinguishing between smooth concordance classes of topologically slice knots. Cochran–Harvey–Horn also defined the bipolar filtration of \mathcal{C} , $\mathcal{B}_n := \mathcal{P}_n \cap \mathcal{N}_n$ [CHH13], and it is expected that this filtration will non-trivially filter topologically slice knots at each n . This is currently known for knots at $n \leq 1$ [CHH13, CH12]. For links of two or more components, this is known for all n by work of Cha–Powell [CP12].

The definitions of the positive and negative filtrations are analogous to that of the n -solvable filtration, in that one filters knots based on certain properties of 4-manifolds within which they bound slice disks. The primary goal of this thesis is to construct the counterparts for the positive and negative filtrations using the first paradigm for approximating sliceness, i.e. by considering geometric objects which approximate disks bounded by knots in B^4 .

1.1.2 Casson towers

Casson towers [Cas86, Fre82]—4-dimensional objects built using layers of immersed disks (see Figure 1.4 for a schematic picture)—form an important piece of this thesis and have been of central importance in the study of topological 4-manifolds. The

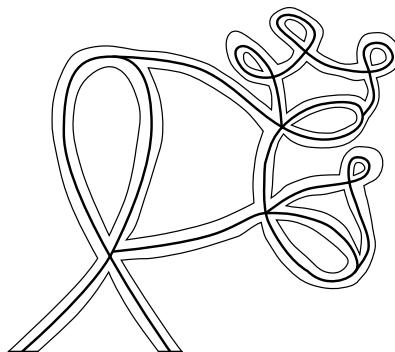


Figure 1.4: Schematic diagram of a Casson tower of height three.

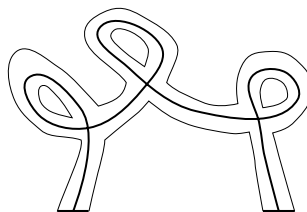


Figure 1.5: Schematic diagram of a regular neighborhood of an immersed disk with three kinks. At each kink, we can see a simple closed curve which cannot be contracted within the disk.

general idea behind them is as follows. Any smooth map of a disk D into a 4-manifold M such that ∂D is mapped to ∂M can be perturbed slightly to get an immersed disk D' . An immersed disk differs from an embedded disk in that there are isolated points of self-intersection, i.e. *kinks*. In particular, while any simple closed curve on an embedded disk can be contracted to a point, there exist curves on immersed disks which cannot: in particular, consider the simple closed curves $\{\alpha_i\}$ which travel from a kink along one sheet of the disk and returns to the kink along the other sheet (see Figure 1.5). The fundamental group of an immersed disk is freely generated by the curves $\{\alpha_i\}$. If we could find embedded disks bounded by these curves in the complement of D' in M , we could use those disks to resolve the kinks and get an embedded disk (using a process known as the *Whitney trick*). However, finding such disks is generally quite hard, and at best we can only hope for a disjoint set of immersed disks $\{D''_i\}$ —one for each α_i . While this would not allow us to obtain an embedded disk, this would nullify each α_i in the union of D and $\{D''_i\}$ since they would now bound disks. However, we do get a new set of fundamental group generators, namely curves at the kinks in the new disks $\{D''_i\}$. Nonetheless, in this new object, we have pushed the generators of the fundamental group to the second stage, and we can repeat the process. That is, starting with an immersed disk, we can attach immersed disks at the curves at each kink, then attach even more kinky disks at the curves at the kinks in the second layer or disks and so on; the object we obtain if we

stop at any finite stage and take a regular neighborhood is called a Casson tower (see Figure 1.4; the fundamental group of a Casson tower is freely generated by curves at the kinks of the disks attached in the terminal layer, for example, there are four generators for the fundamental group of the tower shown in Figure 1.4.

In the 1970's, Casson had the wonderfully bold idea of continuing the process of building a Casson tower indefinitely, i.e. he defined a *Casson handle* to be a Casson tower with infinitely many layers of immersed disks and showed, amazingly, that every Casson handle is proper homotopic, relative to its attaching boundary, to $\mathbb{D}^2 \times \mathbb{R}^2$. This foreshadowed one of the truly mind-boggling results in topology: in 1981, Freedman [Fre82] showed that any Casson handle is in fact *homeomorphic* to $\mathbb{D}^2 \times \mathbb{R}^2$, i.e. any Casson handle is homeomorphic to a regular neighborhood of an embedded disk. In essence, this says that by continuing to push our fundamental group generators farther and farther away, by pushing to infinity we can simply get rid of them. This highly technical result led to a wealth of beautiful results about topological 4-manifolds, such as the topological h -cobordism theorem (which implies the 4-dimensional topological Poincaré Conjecture) and Freedman's complete classification of topological 4-manifolds. Additionally, it implies that any knot with Alexander polynomial 1 is topologically slice. Freedman also proved a Reimbedding Theorem for Casson towers showing that any Casson tower of height six contains a Casson handle within it (this later improved by Gompf in [GS84] who showed that any Casson tower of height five contains a Casson handle).

It is worth noting that Freedman's result does not hold in the smooth category and in fact, there are uncountably many diffeomorphism classes of Casson handles [Gom84, Gom89].

1.2 Results

In this thesis we will prove counterparts of Theorem 1 for the positive and negative filtrations in terms of Casson towers. In particular, we define several new filtrations of \mathcal{C} :

$$\begin{aligned} \cdots \subseteq \mathfrak{C}_{n+1} \subseteq \mathfrak{C}_n \subseteq \cdots \subseteq \mathfrak{C}_1 \subseteq \mathcal{C} \\ \cdots \subseteq \mathfrak{C}_{n+1}^+ \subseteq \mathfrak{C}_n^+ \subseteq \cdots \subseteq \mathfrak{C}_1^+ \subseteq \mathcal{C} \\ \cdots \subseteq \mathfrak{C}_{n+1}^- \subseteq \mathfrak{C}_n^- \subseteq \cdots \subseteq \mathfrak{C}_1^- \subseteq \mathcal{C} \\ \cdots \subseteq \mathfrak{C}_{2,n+1} \subseteq \mathfrak{C}_{2,n} \subseteq \cdots \subseteq \mathfrak{C}_{2,0} \subseteq \mathfrak{C}_1 \subseteq \mathcal{C} \\ \cdots \subseteq \mathfrak{C}_{2,n+1}^+ \subseteq \mathfrak{C}_{2,n}^+ \subseteq \cdots \subseteq \mathfrak{C}_{2,0}^+ \subseteq \mathfrak{C}_1^+ \subseteq \mathcal{C} \\ \cdots \subseteq \mathfrak{C}_{2,n+1}^- \subseteq \mathfrak{C}_{2,n}^- \subseteq \cdots \subseteq \mathfrak{C}_{2,0}^- \subseteq \mathfrak{C}_1^- \subseteq \mathcal{C} \end{aligned}$$

Any knot K that can be changed to a slice knot by only changing positive crossings to negative crossings is known to be in \mathcal{P}_0 by [CHH13, Proposition 3.1] and [CL86, Lemma 3.4]. Such a knot also bounds an immersed disk in B^4 with only positive self-intersections (i.e. kinks). Indeed if a knot K bounds an immersed disk in B^4 with only positive kinks, we can ‘blow up’ the kinks, i.e. connect-sum with a $\mathbb{C}\mathbb{P}(2)$ at each kink, to obtain a slice disk for K in a 4-manifold with positive definite intersection form as called for in the definition for \mathcal{P}_0 . (This reveals how the definition of \mathcal{P}_0 is a generalization of both the ordering on knot concordance classes given by [CG88] and [CL86], and the notion of *kinkiness* of knots defined by Gompf in [Gom86].) Similar statements hold for knots bounding immersed disks with only negative kinks and \mathcal{N}_0 . Since bounding an immersed disk is closely related to membership in \mathcal{P}_0 and \mathcal{N}_0 , Casson towers—built using layers of immersed disks—are natural objects to study in this context.

In this thesis, we establish several relationships between various filtrations of \mathcal{C} (Theorem A) and completely characterize knots in \mathfrak{C}_1^\pm , i.e. knots which bound kinky disks in B^4 with only positive (resp. negative) kinks (Theorem B) as follows.

Theorem A. *Let $\{\mathcal{G}_n\}_{n=0}^\infty$ the (symmetric) grope filtration of \mathcal{C} . $\{\mathcal{G}_{2,n}\}_{n=0}^\infty$ is a slight enlargement of the grope filtration. (Precise definitions for the filtrations can be found in Chapter 2.)*

For any $n \geq 0$,

$$(i) \quad \mathfrak{C}_{n+2} \subseteq \mathcal{G}_{n+2} \subseteq \mathcal{F}_n,$$

$$(ii) \quad \mathfrak{C}_{2,n} \subseteq \mathcal{G}_{2,n} \subseteq \mathcal{F}_n,$$

$$(iii) \quad \mathfrak{C}_{n+2}^+ \subseteq \mathfrak{C}_{2,n}^+ \subseteq \mathcal{P}_n,$$

$$(iv) \quad \mathfrak{C}_{n+2}^- \subseteq \mathfrak{C}_{2,n}^- \subseteq \mathcal{N}_n.$$

Theorem B. *For any knot K , the following statements are equivalent.*

$$(a) \quad K \in \mathfrak{C}_1^+ \text{ (resp. } \mathfrak{C}_1^-)$$

(b) K is concordant to a fusion knot of split positive (resp. negative) Hopf links

(c) K is concordant to a knot which can be changed to a ribbon knot by changing only positive (resp. negative) crossings.

The second inclusion in part (i) of Theorem A is exactly the second result listed earlier in Theorem 1 [COT03, Theorem 8.11] and we only include it here for completeness.

Let \mathcal{W}_n denote the set of knots which bound Whitney towers of height n in B^4 . From their definitions it can be easily seen that any Casson tower yields a Whitney tower with the same attaching curve. As a result, in conjunction with Theorem 1 [COT03, Theorem 8.12], it was already known that $\mathfrak{C}_{n+2} \subseteq \mathcal{W}_{n+2} \subseteq \mathcal{F}_n$ for all n . Our

contribution consists of showing that if a knot bounds a Casson tower T of height n in B^4 , it bounds a properly embedded grope of height n within T (Proposition 3.1). In contrast, Schneiderman has shown that if a knot bounds a properly embedded grope of height n in B^4 , it bounds a Whitney tower of height n in B^4 [Sch06, Corollary 2]. The converse to Schneiderman's statement is not known. In summary, it was previously known that $\mathcal{G}_{n+2} \subseteq \mathcal{W}_{n+2} \subseteq \mathcal{F}_n$ and $\mathfrak{C}_{n+2} \subseteq \mathcal{W}_{n+2} \subseteq \mathcal{F}_n$. We have now shown that $\mathfrak{C}_{n+2} \subseteq \mathcal{G}_{n+2} \subseteq \mathcal{W}_{n+2} \subseteq \mathcal{F}_n$.

We will see that $\mathfrak{C}_n^\pm \subseteq \mathfrak{C}_n$ and $\mathfrak{C}_{2,n}^\pm \subseteq \mathfrak{C}_{2,n}$ for all n , and therefore parts (i) and (ii) of Theorem A imply that $\mathfrak{C}_{n+2}^\pm \subseteq \mathcal{F}_n$ and $\mathfrak{C}_{2,n}^\pm \subseteq \mathcal{F}_n$. Along with [CHH13, Proposition 5.5] which states that $\mathcal{P}_n \subseteq \mathcal{F}_n^{\text{odd}}$ (and $\mathcal{N}_n \subseteq \mathcal{F}_n^{\text{odd}}$), we get the following inclusions for each n . ($\{\mathcal{F}_n^{\text{odd}}\}_{n=0}^\infty$ is a larger filtration than the n -solvable filtration, i.e. $\mathcal{F}_n \subseteq \mathcal{F}_n^{\text{odd}}$ for each n .)

$$\begin{array}{cccccccc}
 \mathcal{F}_n & \subseteq & \mathcal{F}_n^{\text{odd}} & & \mathcal{F}_n & \subseteq & \mathcal{F}_n^{\text{odd}} & & \mathcal{F}_n & \subseteq & \mathcal{F}_n^{\text{odd}} & & \mathcal{F}_n & \subseteq & \mathcal{F}_n^{\text{odd}} \\
 \cup & & \cup & & \cup & & \cup & & \cup & & \cup & & \cup & & \cup \\
 \mathfrak{C}_{n+2}^+ & \subseteq & \mathcal{P}_n & & \mathfrak{C}_{2,n}^+ & \subseteq & \mathcal{P}_n & & \mathfrak{C}_{n+2}^- & \subseteq & \mathcal{N}_n & & \mathfrak{C}_{2,n}^- & \subseteq & \mathcal{N}_n
 \end{array}$$

We state the following corollaries to facilitate easy reference in our proofs and examples. They may be considered to be corollaries of Theorem A or of Theorem 1 along with the fact that Casson towers yield Whitney towers with the same attaching curve.

Corollary 1. *If a knot K lies in \mathfrak{C}_2 , $\text{Arf}(K) = 0$.*

Corollary 2. *If a knot K lies in $\mathfrak{C}_{2,1}$, then K is algebraically slice.*

The above statements follow easily from well-known properties of the n -solvable filtration, namely, any knot in \mathcal{F}_0 has trivial Arf invariant and any knot in \mathcal{F}_1 is algebraically slice [COT03].

Gompf's refinement of Freedman's Reimbedding Theorem for Casson towers [Fre82,

Theorem 4.4][GS84, Theorem 5.1] implies that the filtrations $\{\mathfrak{C}_n\}_{n=1}^\infty$ and $\{\mathfrak{C}_n^\pm\}_{n=1}^\infty$ stabilize at $n = 5$, i.e. $\mathfrak{C}_5 = \mathfrak{C}_6 = \mathfrak{C}_7 = \dots$ and $\mathfrak{C}_5^\pm = \mathfrak{C}_6^\pm = \mathfrak{C}_7^\pm = \dots$. Let \mathcal{T} denote the set of all topologically slice knots. By combining the Reimbedding Theorem with results of Freedman [Fre82, Theorem 1.1] and Quinn [Qui82, Proposition 2.2.4][Gom05, Theorem 5.2] we can see that \mathfrak{C}_5 is *equal* to \mathcal{T} . It appears to be widely believed by experts that \mathfrak{C}_3 is equal to \mathcal{T} .

It is worth noting that while $\mathfrak{C}_5 = \mathcal{T}$, each of \mathfrak{C}_5^\pm is a *proper* subset of \mathcal{T} . This mirrors the fact that the positive/negative filtrations are able to distinguish topologically slice knots while the n -solvable filtration cannot.

As we see above, the $\{\mathfrak{C}_n\}_{n=1}^\infty$ filtration stabilizes at $n = 5$ (or conjecturally at $n = 3$). It is also easy to see that $\mathfrak{C}_1 = \mathcal{C}$, i.e. any knot bounds an immersed disk in B^4 . This indicates that if one is interested in studying smooth concordance classes of topologically slice knots one should focus on these levels. The filtration $\{\mathfrak{C}_{2,n}\}_{n=0}^\infty$ is designed specifically to filter knots within these levels, in particular, between \mathfrak{C}_2 and \mathfrak{C}_3 .

We also see, in Corollary 3.9, that $\mathfrak{C}_3 \subseteq \mathfrak{C}_{2,n}$ and $\mathfrak{C}_3^\pm \subseteq \mathfrak{C}_{2,n}^\pm$ for all $n \geq 0$. From part (i) of Theorem A then,

$$\mathfrak{C}_3 \subseteq \bigcap_{n=0}^{\infty} \mathcal{F}_n.$$

The only presently known elements of $\bigcap_{n=0}^{\infty} \mathcal{F}_n$ are topologically slice knots and it is conjectured that $\bigcap_{n=0}^{\infty} \mathcal{F}_n = \mathcal{T}$. From the above, we can infer that either any knot bounding a Casson tower of height three is topologically slice or there exist knots in $\bigcap_{n=0}^{\infty} \mathcal{F}_n$ which are not topologically slice. Similarly, since

$$\bigcap_{n=0}^{\infty} \mathfrak{C}_{2,n} \subseteq \bigcap_{n=0}^{\infty} \mathcal{F}_n$$

we are led to conjecture that any knot in $\bigcap \mathfrak{C}_{2,n}$ is topologically slice.

By parts (iii) and (iv) of Theorem A,

$$\mathfrak{C}_3^+ \subseteq \bigcap_{n=0}^{\infty} \mathcal{P}_n \text{ and } \mathfrak{C}_3^- \subseteq \bigcap_{n=0}^{\infty} \mathcal{N}_n.$$

This indicates that membership in \mathfrak{C}_3^\pm is a very restrictive condition. For example, the results of [CHH13] show how membership in just the zero'th and first levels of the positive and negative filtrations impose severe restrictions on smooth concordance class. This also reveals that while the positive and negative filtrations have had success in distinguishing concordance classes of topologically slice knots, they cannot be used to distinguish between topologically slice knots in \mathfrak{C}_3^\pm .

1.3 Outline of thesis

We start by stating precise definitions of Casson towers and the various filtrations of \mathcal{C} in Chapter 2. Chapters 3 and 4 consist of the proofs of Theorems A and B respectively; additionally in Chapter 4 we give an overview of various notions of positivity of knots and how membership in \mathcal{P}_0 and \mathfrak{C}_1^+ are related to them. In Chapter 5 we will list various properties of the Casson tower filtrations. We generalize our results to the case of (string) links in Chapter 6.

Notation and definitions

2.1 Casson towers

Suppose $f : D \rightarrow M$ is a smooth self-transverse immersion, where D is a genus zero, oriented 2-manifold, M is an oriented, smooth 4-manifold, and $f^{-1}(\partial M) = \partial D$. We will refer to the points of self-intersection of $f(D)$ as *kinks* and $f(D)$ as being *kinky*. Since M and D are oriented, each kink of $f(D)$ has a canonical sign. A regular neighborhood of a kinky disk in a 4-manifold will be called a *kinky handle*.

In our proofs we will frequently utilize Kirby diagrams to describe 4-manifolds. Background on Kirby diagrams and Kirby calculus can be found in [GS99]. Kirby diagrams for a kinky handle with a single positive kink are given in Figure 2.1, where the sign of the clasp corresponds to the sign of the kink. To obtain pictures for a kinky disk with a single negative kink, we need simply to use the negative clasp. For more details, the interested reader is directed to Chapter 6 of [GS99].

Using kinky handles we may construct a *Casson tower*. Detailed descriptions of Casson towers may be found in [Cas86, Fre82, GS84]. A Casson tower of height one is simply a kinky handle. A Casson tower of height $k + 1$ is obtained from one of height k by attaching kinky handles to the 0-framed meridians of the dotted circles

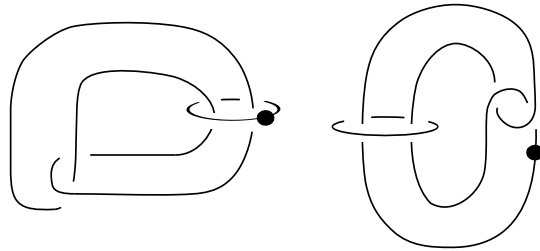


Figure 2.1: Two Kirby diagrams for a kinky handle with a single positive kink. The two panels are pictures of the same space and differ only by an isotopy of curves; we show both versions since each will appear later in the paper. The dotted curve represents a 1–handle, and the other curve is the attaching curve for the kinky handle.

representing 1–handles in the k^{th} level kinky handles, for example, in Figure 2.2. The corresponding infinite construction, i.e. a Casson tower with infinite height is called a *Casson handle*.

A Casson tower has a fixed curve in its boundary, called the *attaching curve*. If we consider the kinky disk D which forms the core of the first level of the Casson tower, the attaching curve is exactly the boundary of D . We will consider every Casson tower to have a fixed decomposition into kinky handles. The meridians of the dotted circles of the last layer of kinky handles, that is, simple loops traversing the terminal 1–handles exactly once, generate the fundamental group of the Casson tower. A set of such meridians will be referred to as the *standard set of curves* for a Casson tower. Sometimes we will also refer to the meridians of the dotted circles at a given stage within a Casson tower. For example, we might refer to the standard set of curves at the second stage of a Casson tower of height four. If a stage of the Casson tower is not specified, we refer to the standard set of curves at the terminal stage. A Kirby diagram for a general Casson tower of height two is shown in Figure 2.2.

Every Casson tower has a 2–complex as a strong deformation retract, called its *core*. For a Casson tower of height one, namely the regular neighborhood of a disk D with transverse self-intersections, the core is exactly the immersed disk D . For a

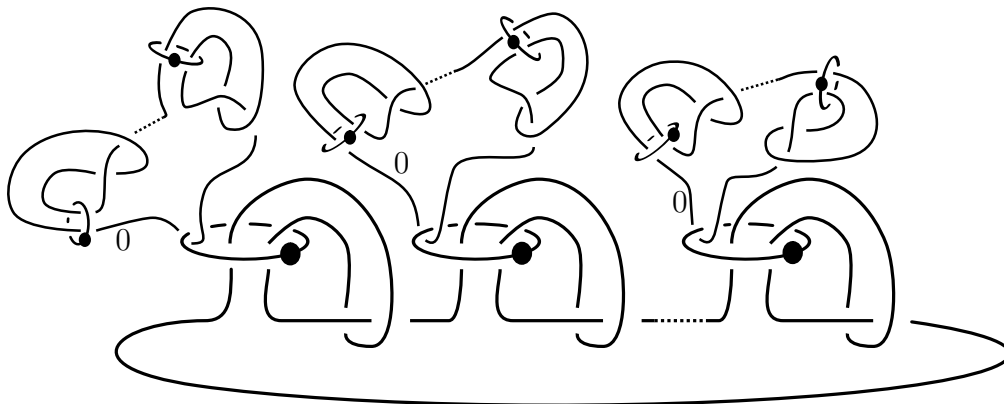


Figure 2.2: A Kirby diagram for a general Casson tower of height two. The bottom-most curve in the picture is the attaching curve.

Casson tower of greater height, the core consists of the cores of each kinky handle along with certain canonical annuli. This is described in greater detail in [GS84, Section 2.2.6].

We will say that a curve γ ‘bounds a Casson tower T in a 4-manifold M ’ if there is a proper embedding of T in M where a 0-framed regular neighborhood of the attaching curve (seen in a Kirby diagram for T) is identified with a 0-framed neighborhood of γ in ∂M . If the 4-manifold is not mentioned, the reader should assume it to be B^4 . In particular, this means that if a knot K is said to bound, say, the Casson tower T shown in Figure 2.2, the 0-framed longitude of K in S^3 can be seen as the 0-framed longitude of the attaching curve of T .

Recall that for any group G , $G^{(n)}$ denotes the n^{th} term of its derived series.

Definition 1. A knot K is said to be in \mathfrak{C}_n if it bounds a Casson tower of height n .

Definition 2. A knot K is said to be in $\mathfrak{C}_{2,n}$ if it bounds a Casson tower T of height two such that each member of a standard set of curves for T is in $\pi_1(B^4 - C)^{(n)}$, where C is the core of T .

Clearly, each \mathfrak{C}_n and $\mathfrak{C}_{2,n}$ is a subgroup of \mathcal{C} with respect to the connected sum

operation on knot concordance classes; in particular, the property of bounding a Casson tower of a particular variety is constant within every knot concordance class, i.e. for example, if K is in \mathfrak{C}_n and J is concordant to K , then J is in \mathfrak{C}_n .

Definition 3. A knot K is said to be in \mathfrak{C}_n^+ (resp. \mathfrak{C}_n^-) if it bounds a Casson tower of height n such that the kinks in the first stage kinky disk are all positive (resp. negative).

Definition 4. A knot K is said to be in $\mathfrak{C}_{2,n}^+$ (resp. $\mathfrak{C}_{2,n}^-$) if it bounds a Casson tower T of height two such that the kinks in the first stage kinky disk are all positive (resp. negative) and each member of a standard set of curves for T is in $\pi_1(B^4 - C)^{(n)}$, where C is the core of T .

Each \mathfrak{C}_n^\pm and $\mathfrak{C}_{2,n}^\pm$ is a monoid with respect to the connected sum operation on knot concordance classes. They are *not* subgroups of \mathcal{C} , since if $K \in \mathfrak{C}_n^+$, $-K \in \mathfrak{C}_n^-$ but $-K$ may not be in \mathfrak{C}_n^+ ; and if $K \in \mathfrak{C}_{2,n}^+$, $-K \in \mathfrak{C}_{2,n}^-$ but $-K$ may not be in $\mathfrak{C}_{2,n}^+$.

We will sometimes use the notation \mathfrak{C}_n^\pm when referring to either of \mathfrak{C}_n^+ or \mathfrak{C}_n^- . Clearly,

$$\cdots \mathfrak{C}_{n+1}^\pm \subseteq \mathfrak{C}_n^\pm \subseteq \cdots \subseteq \mathfrak{C}_1^\pm \subseteq \mathcal{C}$$

$$\cdots \mathfrak{C}_{n+1} \subseteq \mathfrak{C}_n \subseteq \cdots \subseteq \mathfrak{C}_1 \subseteq \mathcal{C}$$

and

$$\cdots \mathfrak{C}_{2,n+1}^\pm \subseteq \mathfrak{C}_{2,n}^\pm \subseteq \cdots \subseteq \mathfrak{C}_{2,1}^\pm \subseteq \mathfrak{C}_{2,0}^\pm \equiv \mathfrak{C}_2^\pm \subseteq \mathcal{C}$$

$$\cdots \mathfrak{C}_{2,n+1} \subseteq \mathfrak{C}_{2,n} \subseteq \cdots \subseteq \mathfrak{C}_{2,1} \subseteq \mathfrak{C}_{2,0} \equiv \mathfrak{C}_2 \subseteq \mathcal{C}$$

Studying the filtrations $\{\mathfrak{C}_n\}_{n=1}^\infty$ is unsatisfying in general since $\mathfrak{C}_5 = \mathfrak{C}_6 = \mathfrak{C}_7 = \cdots$. This is due to Freedman's Reimbedding Theorem [Fre82, Theorem 4.4] (later improved by Gompf–Singh in [GS84, Theorem 5.1]) which states that any Casson tower of height five contains within it arbitrarily high Casson towers sharing its initial

three stages. In particular, this allows us to see that a Casson tower of height five contains a Casson handle within it. Along with Freedman’s extraordinary theorem that any Casson handle is homeomorphic to an open 2–handle [Fre82, Theorem 1.1], this implies that if a knot bounds a Casson tower T of height five, it has a topological slice disk within T itself.

The question of whether a given Casson tower contains a topological slice disk for its attaching curve can be rephrased in terms of whether a certain iterated, ramified Whitehead double of the Hopf link is topologically slice in the 4–ball with standard disks. (This relationship can be easily seen using Kirby diagrams and is indicated in [Kir89, pp. 80–81].) Using this connection it is easy to infer that not all Casson towers of height one or two contain topological disks. The simplest Casson towers of height three and four (i.e. with a single kink at each stage) contain topological slice disks for the attaching curve [Fre88], but this is not known for such towers in general. It appears to be widely believed by experts that all Casson towers of height three and higher contain topological slice disks for the attaching curve¹.

Let \mathcal{T} denote the set of all topologically slice knots. The above shows that if a knot bounds a ‘tall enough’ Casson tower (height five is sufficient, height three is conjectured to be enough), it is topologically slice. That is, $\mathfrak{C}_5 \subseteq \mathcal{T}$. Indeed, a result of Quinn [Qui82, Proposition 2.2.4][Gom05, Theorem 5.2] shows that any topologically slice knot bounds a Casson handle in B^4 . Therefore, \mathfrak{C}_5 is *equal* to \mathcal{T} . If any Casson tower of height three contains a topological slice disk for its attaching curve, \mathfrak{C}_3 would be equal to \mathcal{T} .

¹The current literature is somewhat misleading on the status of this conjecture for general Casson towers of height three and four.

2.2 Filtrations of the knot concordance group

We end this chapter by recalling the definitions of several filtrations of \mathcal{C} .

Definition 2.1 (Definition 2.2 of [CHH13]). For any $n \geq 0$, a knot $K \subseteq S^3$ is in \mathcal{P}_n (resp. \mathcal{N}_n) and is said to be n -positive (resp. n -negative) if there exists a smooth, compact, oriented 4-manifold V such that there is a properly embedded, smooth 2-disk $\Delta \subseteq V$ with $\partial\Delta = K$, $\partial V = S^3$, $[\Delta]$ trivial in $H_2(V, S^3)$ and

1. $\pi_1(V) = 0$
2. the intersection form on $H_2(V)$ is positive definite (resp. negative definite)
3. $H_2(V)$ has a basis represented by a collection of surfaces $\{S_i\}$ disjointly embedded in the exterior of Δ such that $\pi_1(S_i) \subseteq \pi_1(V - \Delta)^{(n)}$ for all i .

Definition 2.2 ([COT03]). For any $n \geq 0$, a knot $K \subseteq S^3$ is in \mathcal{F}_n and is said to be n -solvable if there exists a smooth, compact, oriented 4-manifold V such that there is a properly embedded, smooth 2-disk $\Delta \subseteq V$ with $\partial\Delta = K$, $\partial V = S^3$, $[\Delta]$ trivial in $H_2(V, S^3)$ and

1. $H_1(V) = 0$
2. there exist surfaces $\{L_1, D_1, L_2, D_2, \dots, L_k, D_k\}$ (with product neighborhoods) embedded in $V - \Delta$ which form an ordered basis for $H_2(V)$ such that
 - (a) for each i , L_i and D_i intersect transversely and positively exactly once
 - (b) $L_i \cap D_j$, $L_i \cap L_j$, and $D_i \cap D_j$ are each empty if $i \neq j$
 - (c) $\pi_1(L_i) \subseteq \pi_1(V - \Delta)^{(n)}$ for all i
 - (d) $\pi_1(D_i) \subseteq \pi_1(V - \Delta)^{(n)}$ for all i .

Remark 2.3. The above definition appears different from the original definition of n -solvability in [COT03] at first glance, but the equivalence between the two definitions is straightforward and we refrain from including the proof here. (A proof for the equivalence between the corresponding definitions for the n -positive filtration can be found in [CHH13, Proposition 5.2].)

The original definition of the n -solvable filtration in [COT03] was concerned with the *topological* knot concordance group. Here, as in several recent works in the literature, we are using a version of the filtration for the *smooth* knot concordance group.

If the D_i in the above definition are not required to have product neighborhoods, we get a slight enlargement of the n -solvable filtration, $\{\mathcal{F}_n^{\text{odd}}\}_{n=0}^{\infty}$.

Definition 2.4. A *grope* is a pair (2-complex, attaching circle). A grope of height one is a compact, oriented surface Σ with a single boundary component, *the attaching circle*. Gropes of greater height are defined recursively as follows. Let $\{\alpha_i, \beta_i : i = 1, \dots, g\}$ be disjointly embedded curves representing a symplectic basis for $H_1(\Sigma)$, where Σ is a grope of height one. A grope of height n is obtained by attaching gropes of height $n - 1$ along its attaching circle to each α_i and β_i in Σ .

Remark 2.5. The above gropes are sometimes referred to as ‘symmetric’ gropes and therefore, the following construction is sometimes referred to as the *symmetric grope filtration*.

Definition 2.6 ([COT03]). For any $n \geq 1$, a knot $K \subseteq S^3$ is in \mathcal{G}_n if K extends to a proper embedding of a grope of height n with its untwisted framing in B^4 . This gives the *grope filtration of \mathcal{C}* , $\{\mathcal{G}_n\}_{n=1}^{\infty}$.

Remark 2.7. Note that the above definition differs from our other formulations in that there is an additional ‘framing’ requirement. We could instead include the

framing requirement within the definition of a grope. The resulting 4–dimensional object has been called a ‘Grope’ elsewhere in the literature (for example, in [Hor10]). That is, a knot K is in \mathcal{G}_n if it bounds a Grope in B^4 .

Definition 2.8. For any $n \geq 0$, a knot $K \subseteq S^3$ is in $\mathcal{G}_{2,n}$ if K extends to a proper embedding of a grope G of height two with its untwisted framing in B^4 such that pushoffs of each member of a symplectic basis for the first homology groups of the second stage surfaces of G are in $\pi_1(B^4 - G)^{(n)}$.

Remark 2.9. The groups $\mathcal{G}_{2,n}$ defined above have not appeared in the literature before to the author’s knowledge. However, several proofs of results related to the grope filtration hold for the filtration $\{\mathcal{G}_{2,n}\}_{n=0}^\infty$; this is perhaps unsurprising since it is easily seen that $\mathcal{G}_{n+2} \subseteq \mathcal{G}_{2,n}$ for each n . The following is an example of such a result.

Theorem 2.10 (Theorem 8.11 of [COT03]). $\mathcal{G}_{n+2} \subseteq \mathcal{G}_{2,n} \subseteq \mathcal{F}_n$ for each n .

Proof. Suppose a knot K bounds a grope G in B^4 . If a curve on the second stage surfaces bounds a grope of height n away from the first two stages (call it G'), the curve lies in $\pi_1(B^4 - G')^{(n)}$; as a result the first inclusion is clear.

The second inclusion follows very easily from a close reading of the proof of [COT03, Theorem 8.11] (Theorem 1) where they show that $\mathcal{G}_{n+2} \subseteq \mathcal{F}_n$. Briefly, given a grope G of height $n+2$ bounded by a knot, they only use the first two stages (call it G') and the fact that a symplectic basis for H_1 of each second stage surface is in $\pi_1(B^4 - G')^{(n)}$. □

Proof of Theorem A

In this chapter we prove several results connecting the types of Casson towers bounded by a knot K and membership within the many filtrations of \mathcal{C} . Together these results comprise Theorem A.

Proposition 3.1. *The attaching curve of a Casson tower T of height n bounds a properly embedded grope of height n within T .*

Proof. A simple case is pictured in Figure 3.1, showing a neighborhood of the first two stages of a Casson tower with a single kink in each stage. We will directly and explicitly construct a grope bounded by the attaching curve (the leftmost curve pictured), which will show how to proceed in the general case. The first stage of the grope, Σ , is shown in Figure 3.2.

It is easy to see, abstractly, that both the meridian m and the longitude ℓ of the first stage are homotopic to α_1 , the meridian of the dotted circle. We easily tube ‘inside Σ ’ from m to α_1 , as shown in Figure 3.3. We also see an embedded annulus, shown in Figure 3.3, cobounded by ℓ and a pushoff of α_1 . These two annuli intersect exactly once (as desired) at the point of intersection of m and ℓ .

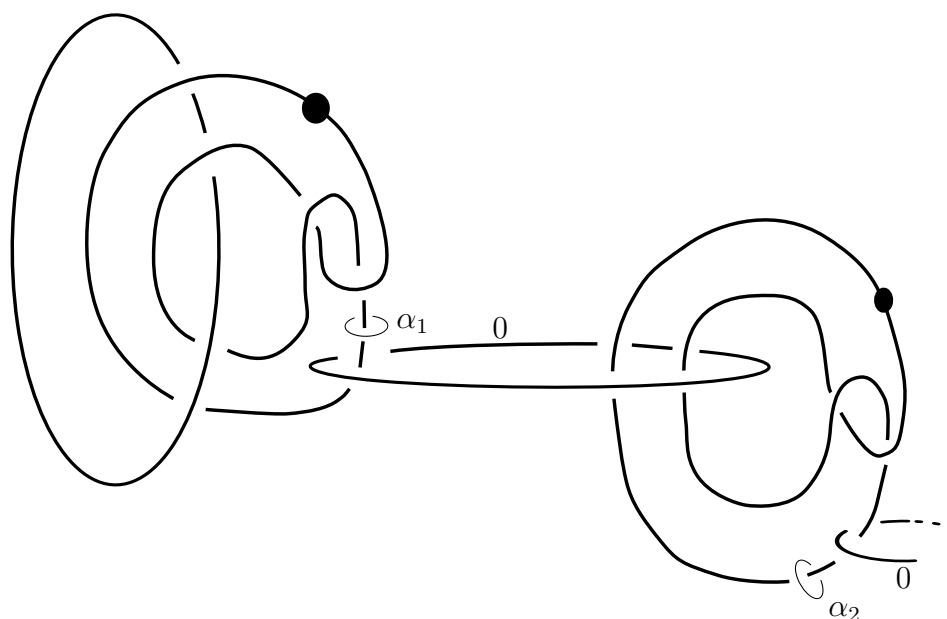


Figure 3.1: Proof of Proposition 3.1: A Kirby diagram for the first two stages of a Casson tower with a single kink at each stage.

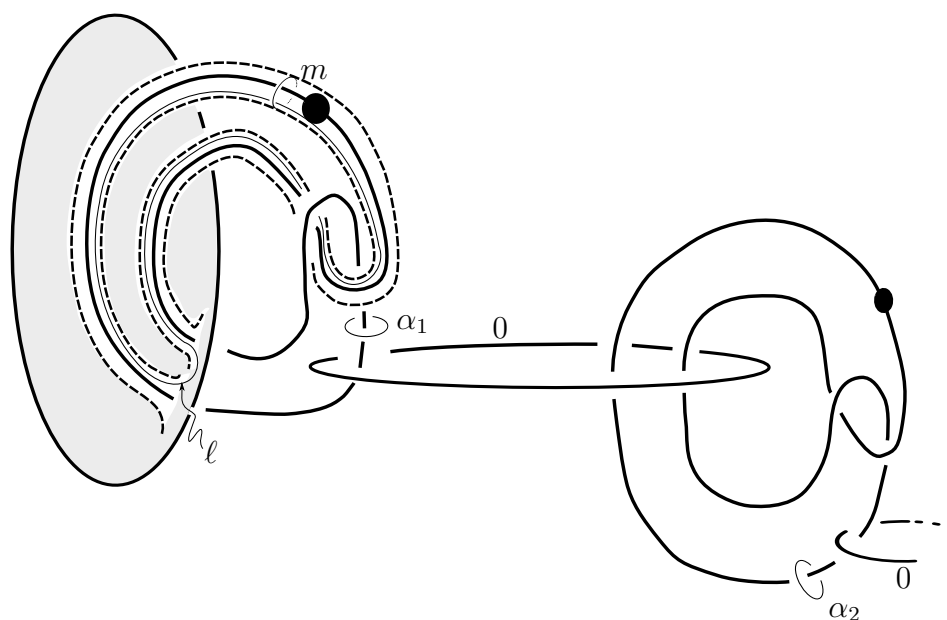


Figure 3.2: Proof of Proposition 3.1: Σ , the first stage of the grope, consists of the ‘obvious’ disk bounded by the attaching curve with a tube (dashed) along the dotted circle. m and l denote the meridian and longitude respectively.

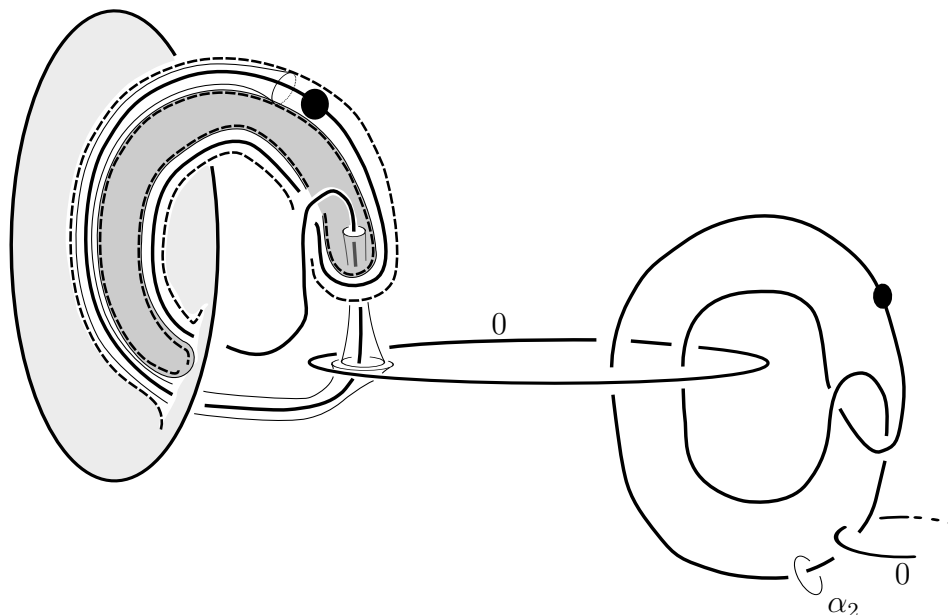


Figure 3.3: Proof of Proposition 3.1: Surfaces connecting the meridian and longitude of Σ to pushoffs of the meridian of the first-level dotted circle.

α_1 and a pushoff of α_1 bound disjoint surfaces in the complement of Σ , as follows. Each surface consists of the core (or a pushoff of the core) of the attached 0-framed 2-handle tubed along the next dotted circle, as shown in Figure 3.4. Since the 2-handle is attached with 0-framing, the pushoffs do not intersect. These surfaces, along with the annuli between m and α_1 , and ℓ and α_1 , form the second stage of our grope.

Constructing the third stage surfaces of the grope will indicate how to proceed in subsequent stages. As before, we have various meridians and longitudes which are abstractly homotopic to the meridian of the second dotted circle, α_2 . We will construct disjoint annuli cobounded by these curves and pushoffs of α_2 , the meridian of the second stage dotted circle, away from the first two stages. We can proceed exactly as we did before to obtain annuli that are disjoint from each other, but since the second stage surfaces are ‘nested’, (most of) the annuli intersect the second stage surfaces. However, these intersections are particularly nice—they are boundary-parallel circles in the annuli. We can push these intersections into the 4-ball to

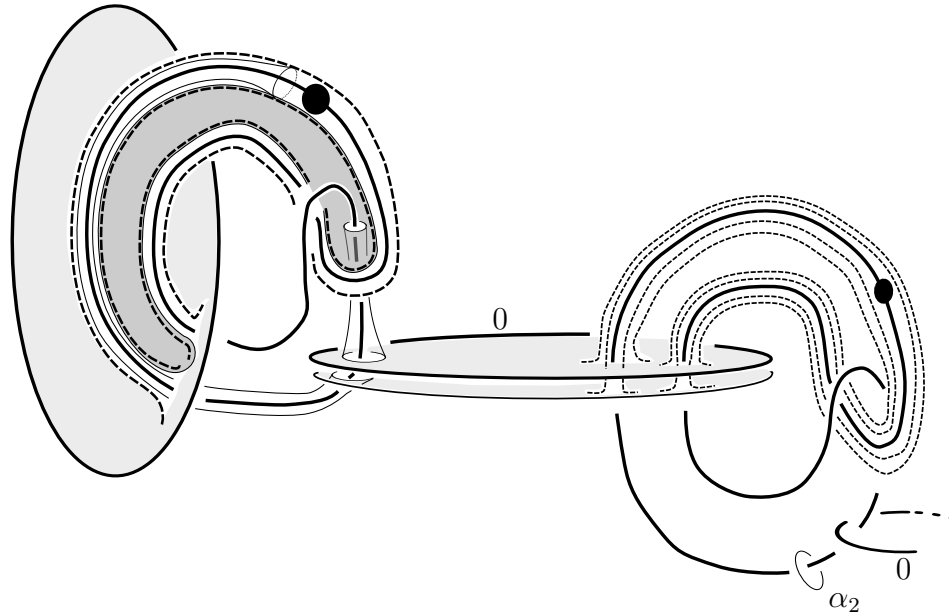


Figure 3.4: Proof of Proposition 3.1: The second stage surfaces of the grope use the annuli constructed previously (in Figure 3.3) and the 0-framed 2-handle attached to the meridian of the first stage dotted circle. We use two copies of the core of the attached handle in addition to the ‘obvious’ disk shaded gray in the picture, with a tube about the second stage dotted circle. Notice that two copies of the tubes are needed and they are nested.

get disjoint annuli. (Here is a good toy analogy. Consider two nested, standard, unknotted tori in S^3 . Any meridional disk of the outer torus will intersect the inner torus in a circle, but we can push the disk into the 4-ball in a neighborhood of the circle to get a meridional disk for the outer torus which is disjoint from the inner torus and still ‘mostly’ in S^3 .) Since α_2 and its pushoffs bound disjoint surfaces as before, we can construct all subsequent stages of the grope. It is easy to see, since most of the grope is in 3-dimensional space, that the attaching curve bounds this grope with untwisted framing.

If we had started with a more complicated Casson tower, we would have obtained a more complicated grope through an identical process. The genus of the first stage surface is equal to the number of kinks in the base-level kinky handle. \square

The following corollary is immediate.

Corollary 3.2. *For each $n \geq 1$, $\mathfrak{C}_n \subseteq \mathcal{G}_n$.*

Corollary 3.3. *Let \mathcal{T} denote the set of all topologically slice knots. Then,*

$$\mathcal{T} \subseteq \bigcap_{n=1}^{\infty} \mathcal{G}_n.$$

Proof. This follows immediately from Proposition 3.1 and Quinn's result that any topological slice disk for a topologically slice knot contains a Casson handle [Qui82, Proposition 2.2.4][Gom05, Theorem 5.2]. \square

The above was previously known (without using Casson handles). Briefly, a topological slice disk for a knot K is a topologically embedded locally flat grope of arbitrary height. Such a grope can be deformed to yield a smooth grope of arbitrary height (some more detail may be found in [Cha14, Remark 2.19]).

Corollary 3.4. *$\mathfrak{C}_{n+2} \subseteq \mathfrak{C}_{2,n}$ and $\mathfrak{C}_{n+2}^{\pm} \subseteq \mathfrak{C}_{2,n}^{\pm}$ for all $n \geq 0$.*

Proof. Each member of a standard set of curves for the second stage of a Casson tower of height $n + 2$ bounds a Casson tower of height n away from the first two stages. Therefore, by Proposition 3.1, each such curve bounds a grope of height n away from the first two stages. \square

Proposition 3.5. *$\mathfrak{C}_{2,n} \subseteq \mathcal{G}_{2,n} \subseteq \mathcal{F}_n$ for all $n \geq 0$.*

Proof. The second inclusion is from Theorem 2.10. For the first inclusion, suppose we have a knot K in $\mathfrak{C}_{2,n}$. That is, K bounds a Casson tower $T \subseteq B^4$ of height two such that the standard set of curves are in $\pi_1(B^4 - C)^{(n)}$, where C is the core of T . By Proposition 3.1, we know that K bounds a grope G of height two within T . In fact, we see that the generators of the first homology groups of the second stage surfaces

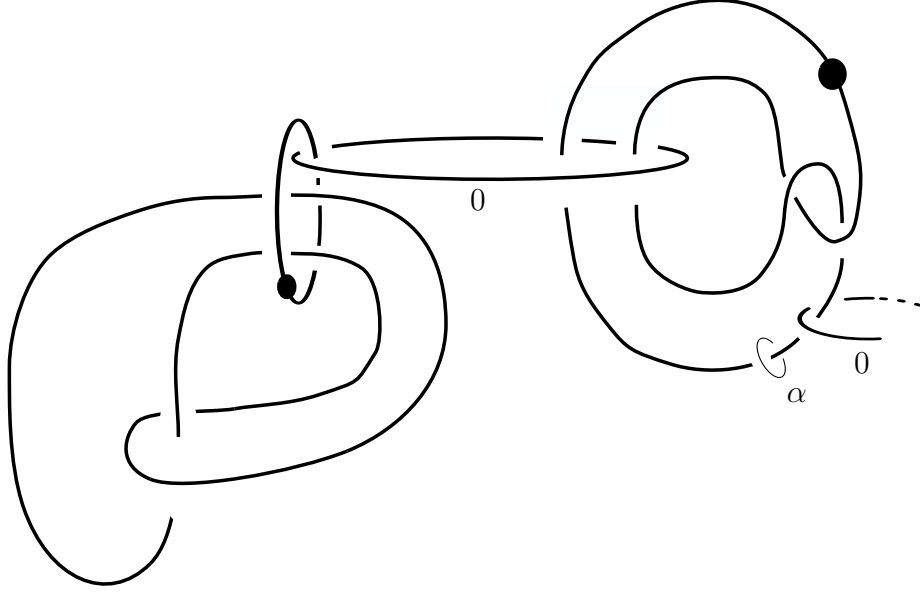


Figure 3.5: Proof of Proposition 3.6: A Kirby diagram of the first two stages of a Casson tower with a single positive kink at each stage.

for G are exactly the meridians of the dotted circles of the second stage kinky disks of T , i.e. they are exactly the standard set of curves for T , which are given to be in $\pi_1(B^4 - C)^{(n)}$. Therefore, $K \in \mathcal{G}_{2,n}$. \square

Proposition 3.6. $\mathfrak{C}_{n+2}^+ \subseteq \mathcal{P}_n$ for all $n \geq 0$. Similarly, $\mathfrak{C}_{n+2}^- \subseteq \mathcal{N}_n$ for all $n \geq 0$.

Proof. As before, we show the proof in the case where there is a single kink at each stage of the Casson tower. The general case will follow analogously.

Figure 3.5 shows a Kirby diagram for the first two stages of a Casson tower T with a single positive kink at each stage. We blow up at the kink in the first stage disk. In our Kirby diagram, Figure 3.6, this introduces a $+1$ -framed 2 -handle, indicating that the new manifold is diffeomorphic to $T \# \mathbb{C}\mathbb{P}(2)$. Since the blow up occurred in the interior of T , we have an embedding $T \# \mathbb{C}\mathbb{P}(2) \hookrightarrow B^4 \# \mathbb{C}\mathbb{P}(2)$ (where earlier we had an embedding $T \hookrightarrow B^4$). Let V denote $B^4 \# \mathbb{C}\mathbb{P}(2)$. A slice disk Δ for K is obvious in $T \# \mathbb{C}\mathbb{P}(2)$, shown shaded in gray in the figure (the attaching curve for the 2 -handle pierces through it twice transversely). Since V is simply connected with

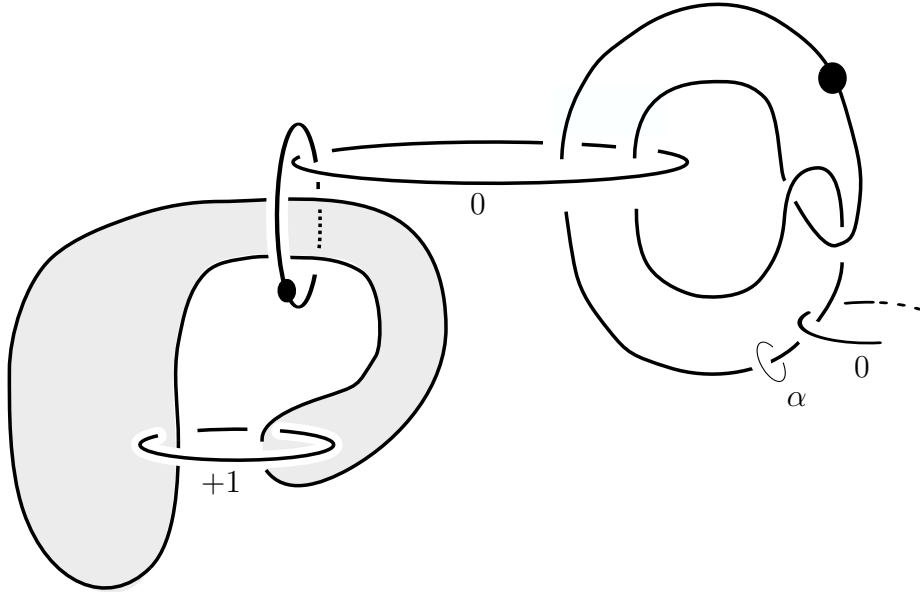


Figure 3.6: Proof of Proposition 3.6: After blowing up the first stage kink, an ‘obvious’ slice disk for the attaching curve can be seen.

positive definite intersection form all that remains to be done is to find a generator S for $H_2(V) \cong \mathbb{Z}$ such that $\pi_1(S) \subseteq \pi_1(V - \Delta)^{(n)}$. We will do so by finding a generator S such that the generators of $\pi_1(S)$ bound gropes of height n in $(T\#\mathbb{C}\mathbb{P}(2)) - \Delta$.

For clarity, we describe how we obtain such an S in several steps. The naïve choice of generator for $H_2(V)$ is the core of the attached $+1$ -framed 2 -handle along with the obvious disk bounded by it, shaded in gray in Figure 3.7. However, this clearly intersects Δ . We can avert this problem by tubing along the attaching curve. While this does yield a torus generator for $H_2(V)$ disjoint from Δ , one of its H_1 -generators is the meridian of the attaching circle (and therefore the knot!) We try to fix this by surgering along the longitude of the torus using the obvious disks (pierced through by the dotted circle in the diagram). The 2 -sphere obtained intersect the dotted circle and so we tube along it, as shown in Figure 3.8. This yields another torus generator for $H_2(V)$, but once again, one of its H_1 -generators is clearly the meridian of the attaching circle. Fortunately, we can address this easily by noting that the meridian

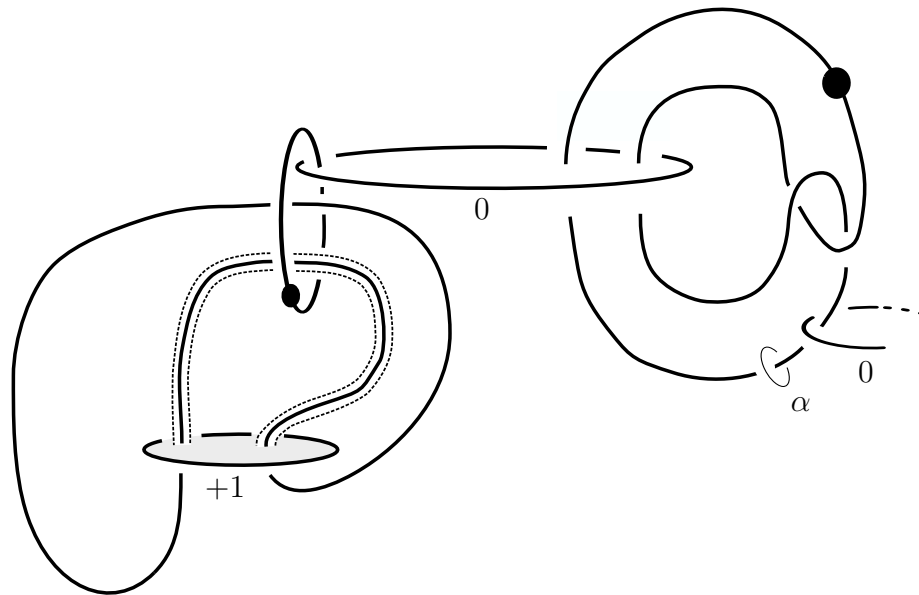


Figure 3.7: Proof of Proposition 3.6: Finding generators of $H_2(V)$.

of the present torus bounds a punctured torus. By cutting along the meridian and gluing in two copies of the punctured torus, we finally obtain a generator S of $H_2(V)$ in Figure 3.9. We claim that this is the desired surface generating $H_2(V)$.

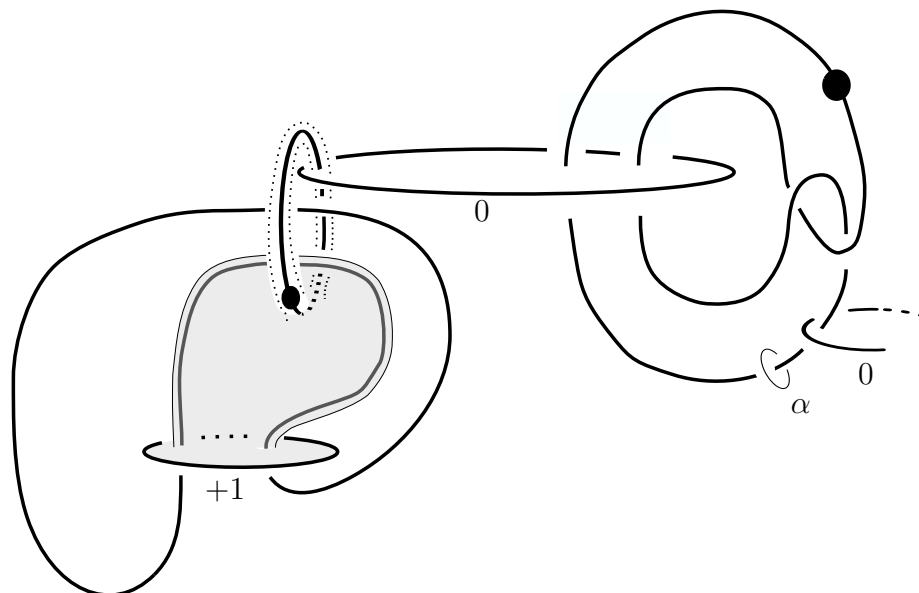


Figure 3.8: Proof of Proposition 3.6: Finding generators of $H_2(V)$.

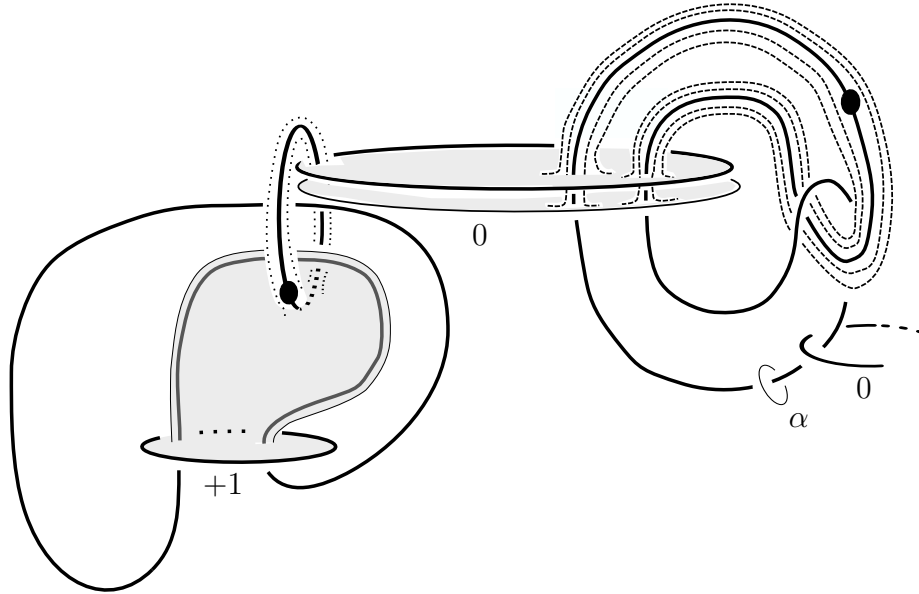


Figure 3.9: Proof of Proposition 3.6: Finding generators of $H_2(V)$.

Note that each member of the standard generating set for $\pi_1(S)$ is homotopic to a meridian of the second stage dotted circle, α (i.e. the standard curve for the second stage of T) away from Δ . Since the standard curve bounds a Casson tower of height n (and therefore a grope of height n) away from Δ , we see that $\pi_1(S) \subseteq \pi_1(V - \Delta)^{(n)}$. But we can do better—we can show that the members of the standard generating set for $\pi_1(S)$ themselves bound disjoint gropes away from the first two stages of T . The only additional step needed is to find disjoint annuli connecting the generators of $\pi_1(S)$ to α . This is the same construction as in the proof of Proposition 3.1 when we constructed the third stage surfaces of a grope, and we omit it to avoid repetition.

The reader might ask why S constructed above is a generator of $H_2(V)$. To see this, start with the naïve choice of generator s , namely, the ‘obvious’ disk (shaded in gray in Figure 3.7) capped off with the core of the $+1$ -framed 2-handle. Take a pushoff \bar{s} of s . \bar{s} and s intersect exactly once transversely with positive sign. Now perform the various tubing operations described above on \bar{s} —we can do so in the complement of s . The resulting surface S will continue to have a single positive

transverse intersection with s and therefore is in the same homology class as s .

For more complicated Casson towers, we apply the same process. The number of generators of $H_2(V)$ needed is equal to the number of kinks in the first stage of the tower. For each kink in the first stage, the genus of the corresponding member of the set of generators of $H_2(V)$ is equal to the number of kinks in the associated second stage kinky disk.

The above shows that $\mathfrak{C}_{n+2}^+ \subseteq \mathcal{P}_n$. For a knot $K \in \mathfrak{C}_{n+2}^-$ the kinks in the first stage kinky disk would be negative and we would blow up using -1 -framed 2 -handles, indicating a connected sum with $\overline{\mathbb{C}\mathbb{P}(2)}$. The rest of the construction is analogous. \square

Proposition 3.7. $\mathfrak{C}_{2,n}^+ \subseteq \mathcal{P}_n$ for all $n \geq 0$. Similarly, $\mathfrak{C}_{2,n}^- \subseteq \mathcal{N}_n$ for all $n \geq 0$.

Proof. The proof of our previous proposition did not truly require the Casson tower beyond the first two levels. If the standard set of curves of a tower of height two is known to be in the n^{th} -derived subgroup of the fundamental group of the exterior of the core of the first two stages, the remainder of the proof follows identically. \square

Corollary 3.8. $\mathfrak{C}_3 \subseteq \mathfrak{C}_{2,n}$ for all n . Similarly, $\mathfrak{C}_3^+ \subseteq \mathfrak{C}_{2,n}^+$ and $\mathfrak{C}_3^- \subseteq \mathfrak{C}_{2,n}^-$ for all n .

Proof. Suppose a knot bounds a Casson tower T of height three. Each member of a standard set of curves for the second stage of T bounds a kinky disk away from C , the core of the first two stages. Therefore, the curves must be null-homotopic away from C and as a result, contained in $\pi_1(B^4 - C)^{(n)}$ for all n . \square

The following is now an immediate corollary of the above result and Proposition 3.7, and reveals the inefficacy of Proposition 3.6 in studying the positive and negative filtrations of \mathcal{C} .

Corollary 3.9. $\mathfrak{C}_3^+ \subseteq \bigcap_{n=0}^{\infty} \mathcal{P}_n$. Similarly, $\mathfrak{C}_3^- \subseteq \bigcap_{n=0}^{\infty} \mathcal{N}_n$.

Proof of Theorem B

Knots in \mathfrak{C}_1^\pm can be completely characterized by the following theorem.

Theorem B. *For any knot K , the following statements are equivalent.*

- (a) $K \in \mathfrak{C}_1^+$ (resp. \mathfrak{C}_1^-)
- (b) K is concordant to a fusion knot of split positive (resp. negative) Hopf links
- (c) K is concordant to a knot which can be changed to a ribbon knot by changing only positive (resp. negative) crossings.

Remark 4.1. In [CL86, Remark 3.3, Lemma 3.4], Cochran–Lickorish showed that if a knot can be changed to the unknot by only changing positive (resp. negative) crossings, it bounds a kinky disk in the 4–ball with only positive (resp. negative) kinks—very little further insight is needed to prove the more general statement (c) \Rightarrow (a). We include it here for completeness.

This result should also be compared with a characterization of knots in a particular subset of \mathcal{P}_0 given by Cochran–Tweedy in [CT].

Proof of Theorem B. Suppose $K \in \mathfrak{C}_1^+$, i.e. K bounds a kinky disk $\bar{\Delta}$ in B^4 with all kinks positive. As before we blow up each kink of $\bar{\Delta}$ with a $\mathbb{C}\mathbb{P}(2)$ to resolve the

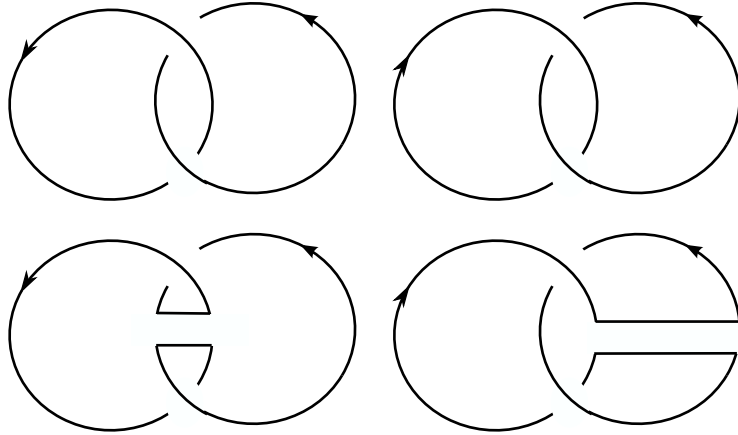


Figure 4.1: Both the positive (left) and the negative (right) Hopf link can be fused to yield the unknot.

singularities of $\overline{\Delta}$. Remove a tubular neighborhood of the core $\mathbb{C}\mathbb{P}(1)$ within each added $\mathbb{C}\mathbb{P}(2)$. This results in a number of additional S^3 boundary components which intersect $\overline{\Delta}$ in positive Hopf links. We can tube these newly created S^3 's together. Since the tube acts like a 1-dimensional submanifold of a 4-manifold, it may be considered to be disjoint from $\overline{\Delta}$. We excise the tube; the resulting 4-manifold W is diffeomorphic to $S^3 \times [0, 1]$. By throwing away any additional components, we get a smooth genus zero surface Δ cobounded by K and a split collection of positive Hopf links. (The Hopf links are split in the sense that they can be separated from one another by a collection of disjoint, smoothly embedded 2-spheres.)

By an isotopy relative to the boundary, we can ensure that the height function on $W \cong S^3 \times [0, 1]$ is Morse with respect to Δ and that the maxima occur at the $t = \frac{1}{5}$ level, the join saddles at the $t = \frac{2}{5}$ level, the split saddles at $t = \frac{3}{5}$ and the minima at $t = \frac{4}{5}$. The intersection of Δ with $t = \frac{1}{2}$ is then a connected 1-manifold embedded in $S^3 \times \{\frac{1}{2}\} \cong S^3$. Call this knot J . The portion of Δ in $S^3 \times [0, \frac{1}{2}]$ gives a concordance between K and J . We will show that J is a fusion of split positive Hopf links.

The portion of Δ in $S^3 \times [\frac{1}{2}, 1]$ is almost what we need already. In particular, it demonstrates J as a fusion of an unlink (from the minima of Δ) and a split collection

of positive Hopf links. However, each component of the unlink can be considered as a fusion of a positive Hopf link, as shown in Figure 4.1. To be more specific, we can use an arc disjoint from Δ to extend each minimum down to $S^3 \times \{\epsilon\}$. Since the minima form an unlink we can keep them split from one another and the Hopf links. Within $S^3 \times [\epsilon, 1]$, we can use saddles to split the unknotted components into positive Hopf links. This shows that J is a fusion of a collection of split positive Hopf links, and therefore (a) \Rightarrow (b).

Now suppose that K is concordant to a fusion knot of split positive Hopf links. Since a positive Hopf link can be changed to an unlink by changing a positive crossing, (b) \Rightarrow (c) is clear.

Suppose that K is concordant to a knot which can be changed to a ribbon knot by only changing positive crossings, i.e. there is a kinky annulus in $S^3 \times [0, 1]$, with only positive kinks, cobounded by K and a ribbon knot J . By appending the slice disk for J , we get a kinky disk with only positive kinks bounded by K in B^4 .

The corresponding statements for \mathfrak{C}_1^- can be proved by an entirely analogous argument. □

Using an almost entirely identical argument, we can prove the following proposition.

Proposition 4.2. *For any knot K , the following statements are equivalent.*

- (a) *K bounds a kinky disk with p positive and n negative kinks.*
- (b) *K is concordant to a fusion knot of p positive Hopf links, n negative Hopf links and an unlink*
- (c) *K is concordant to a knot that can be changed to a ribbon knot by changing p positive and n negative crossings.*

4.1 Positivity of knots

Theorem B involves several notions which might reasonably be referred to as ‘positivity’ for knots. It is instructive to study how they are related to other such notions which are well-established in the literature. Let us start by listing some of these concepts.

1. K is the closure of a positive braid
2. K has a projection where all crossings are positive
3. K is strongly quasipositive
4. K is quasipositive
5. $\kappa_-(K) = 0$
6. K bounds a kinky disk in B^4 with only positive kinks, i.e. $K \in \mathfrak{C}_1^+$
7. K is concordant to a knot that can be changed to a slice knot by changing only positive crossings
8. K is concordant to a fusion knot of a split collection of positive Hopf links
9. $K \in \mathcal{P}_0$

In the list above, κ_- denotes ‘negative kinkiness’, a smooth concordance invariant defined by Gompf in [Gom86]. The terms quasipositivity and strong quasipositivity are due to Rudolph; see [Rud05] for a thorough exposition.

The known relationships between the above notions of positivity of knots are summarized in Figure 4.2. (1) \Rightarrow (2) trivially, but there are examples of knots with positive projections which are not closures of positive braids. Rudolph showed in [Rud99] that knots with positive projections are strongly quasipositive. Strongly

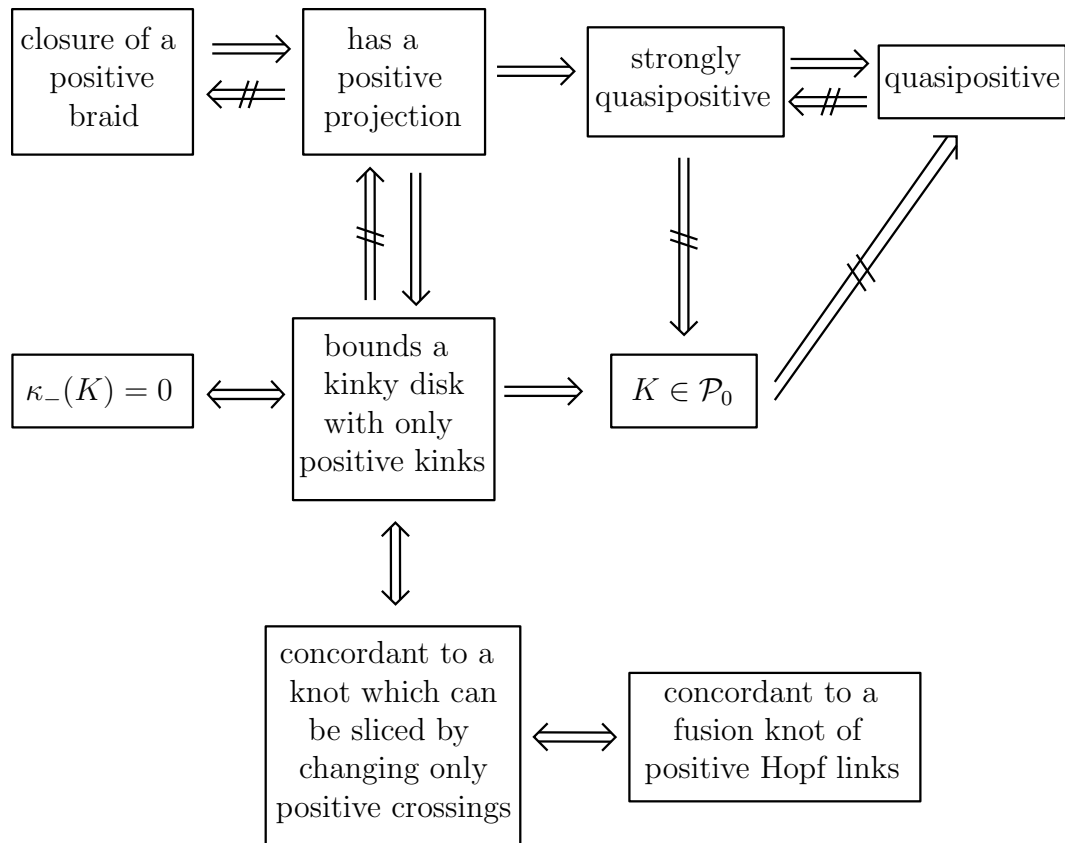


Figure 4.2: Known relationships between some notions of positivity of knots.

quasipositive knots are obviously quasipositive by definition. However, Baader showed in [Baa05] that there exist quasipositive knots that are not strongly quasipositive¹.

(5) and (6) are equivalent by the definition of κ_- and (6), (7) and (8) are equivalent by Theorem B. (5) implies (9) as discussed previously, by ‘blowing up’ at the kinks of a kinky disk. Any knot with a positive projection bounds a kinky disk with only positive kinks; this is so since it can be unknotted by changing only positive crossings. Therefore, (2) implies (6). However, it is known that knots with positive projections necessarily have (strictly) negative signatures [CG88, Prz89, Tra88] while knots (such as the figure eight knot) with zero signature may bound kinky disks with only positive kinks. As a result, (6) does not imply (2).

¹These examples were pointed out by Steven Sivek in response to a question posed by the author on MathOverflow [Siv13].

Rudolph showed that one can construct a strongly quasipositive knot with any given Seifert pairing [Rud83, Rud05]. This implies that we can find strongly quasipositive knots with positive signature, which obstructs membership in \mathcal{P}_0 by [CHH13, Proposition 1.2]. Membership in \mathcal{P}_0 does not imply strong quasipositivity, or even quasipositivity. As pointed out in [Rud89, Remark 4.6], a non-slice knot which is its own mirror image (such as the figure eight knot, which lies in \mathcal{P}_0) cannot be quasipositive. On the other hand, it is true that if K is strongly quasipositive, then $K \notin \mathcal{N}_0$, as follows. Livingston proved in [Liv04]¹ that if K is strongly quasipositive, then $g(K) = g_4(K) = \tau(K)$, where g_4 denotes smooth 4-genus and τ denotes Ozsváth–Szabó [OS03] and Rasmussen’s [Ras03] smooth concordance invariant. Therefore, any non-trivial, strongly quasipositive K has $\tau(K) > 0$, which obstructs membership in \mathcal{N}_0 by [CHH13, Proposition 1.2]. Collectively this paragraph addresses a question posed in Section 3 of [CHH13], seeking the relationship between strong quasipositivity and \mathcal{P}_0 .

The relationships summarized in Figure 4.2 lead to the natural question of whether membership in \mathcal{P}_0 implies any of the equivalent notions (5)–(8). This seems unlikely to be true, but we do not have a counterexample at present.

¹Livingston’s result is not stated in terms of strong quasipositivity. The equivalence of Livingston’s conditions and strong quasipositivity is pointed out by Hedden in the introduction to [Hed10]

Examples and properties

Example 5.1. It is well-known that any knot can be changed to the unknot by changing crossings (the minimum number of crossings that need to be changed is the *unknotting number* of a knot). By tracing the homotopy corresponding to the crossing changes, we see that *every* knot bounds a kinky disk if we impose no restrictions on the signs of the kinks, i.e. every knot lies in \mathfrak{C}_1 .

Example 5.2. Theorem B shows that membership in \mathfrak{C}_1^\pm is harder. From Proposition 1.2 in [CHH13] we know that the signs of various well-known concordance invariants obstruct membership in \mathcal{P}_0 and \mathcal{N}_0 . Since $\mathfrak{C}_1^+ \subseteq \mathcal{P}_0$ and $\mathfrak{C}_1^- \subseteq \mathcal{N}_0$, they also obstruct membership in \mathfrak{C}_1^+ and \mathfrak{C}_1^- . For example, if $\tau(K) < 0$, $K \notin \mathcal{P}_0$, and therefore, $K \notin \mathfrak{C}_1^+$. Similarly, $K \notin \mathfrak{C}_1^+$ if the Levine–Tristram signature of K is strictly positive, or $s(K) < 0$. Using Theorem B, we can then see that the signs of these invariants also obstruct when a knot can be changed to a slice knot by changing only positive or negative crossings. Results of this nature were proved by Cochran–Lickorish and Bohr in [CL86] and [Boh02] respectively.

Example 5.3. By Theorem B, any knot which can be changed to a slice knot by changing positive (resp. negative) crossings lies in \mathfrak{C}_1^+ (resp. \mathfrak{C}_1^-). In particular this

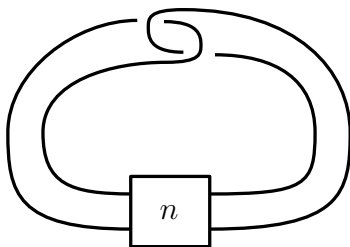


Figure 5.1: The twist knots T_n^- . The box with a number ‘ n ’ inside should be interpreted as n full twists.

implies that any knot with unknotting number (or even *slicing number*) one, lies in either \mathfrak{C}_1^+ or \mathfrak{C}_1^- .

Let T_n^+ denote the positively-clasped twist knots with n twists and T_n^- the negatively-clasped twist knots (see Figure 5.1). Clearly, each T_n^\pm can be unknotted by changing one of the crossings at the clasp and therefore, $T_n^+ \in \mathfrak{C}_1^+$ and $T_n^- \in \mathfrak{C}_1^-$ for all n . On the other hand, for positive n , the knot T_n^+ can be unknotted by changing n negative crossings (undoing the n twists) and therefore, $T_n^+ \in \mathfrak{C}_1^+ \cap \mathfrak{C}_1^-$ for positive n . Similarly, $T_n^- \in \mathfrak{C}_1^+ \cap \mathfrak{C}_1^-$ for negative n . Note that it is easy to see that T_n^+ is a fusion of a positive Hopf link. However, since $T_n^+ \in \mathfrak{C}_1^-$ for positive n , such a knot must also be concordant to a fusion of negative Hopf links by Theorem B—an example is shown in Figure 5.2.

Example 5.4. Example 4.5 of [CHH13] shows that $\text{Wh}_0^-(LHT) \notin \mathcal{P}_0$, where LHT

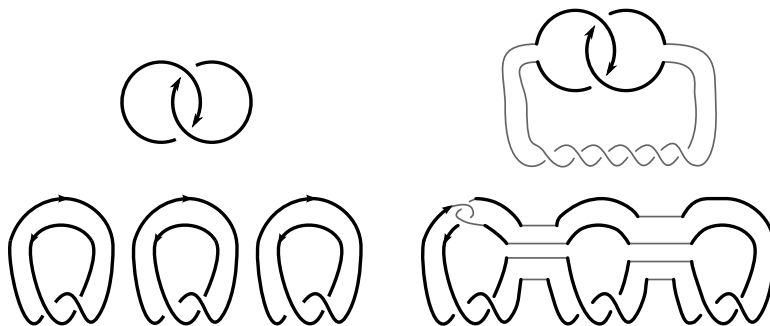


Figure 5.2: The knot T_3^+ can be obtained as a fusion of a single positive Hopf link, or as a fusion of three negative Hopf links. Fusion bands are shown in gray.

is the left-handed trefoil and $\text{Wh}_0^-(\cdot)$ denotes the negatively clasped zero-twisted Whitehead double. Similarly Example 4.6 in [CHH13] shows that if $p < 0$, $q > 0$, and $r > 0$ are odd and $pq + qr + rp = -1$, then the pretzel knots $K(p, q, r) \notin \mathcal{P}_0$. Therefore, since $\mathfrak{C}_1^+ \subseteq \mathcal{P}_0$, none of these knots can bound a kinky disk with only positive kinks and by Theorem B none of these knots can be changed to a slice knot by changing only positive crossings.

Example 5.2 showed that \mathfrak{C}_1^+ and \mathfrak{C}_1^- have non-trivial intersection. However, they are distinct sets, as we see below.

Proposition 5.5. $\mathfrak{C}_1^+ \neq \mathfrak{C}_1^-$.

Proof. Corollary 2 of [Boh02] shows that if K is concordant to a non-trivial strongly quasipositive knot, then $\kappa_+(K) > 0$. This implies that $K \notin \mathfrak{C}_1^-$. However, several strongly quasipositive knots are in \mathfrak{C}_1^+ . For instance, any knot which is a closure of a positive braid (and therefore contained in \mathcal{P}_0) is strongly quasipositive. In fact, Rudolph showed that any knot with a positive projection is strongly quasipositive [Rud99]. This shows that all knots with positive projections are in $\mathfrak{C}_1^+ - \mathfrak{C}_1^-$.

Alternatively, Gompf showed that there exist non-trivial knots with kinkiness $(0, n)$, with $n \neq 0$ in [Gom86]. These knots are clearly in $\mathfrak{C}_1^- - \mathfrak{C}_1^+$. \square

There also exist knots which are is neither \mathfrak{C}_1^+ nor \mathfrak{C}_1^- , as follows. Recall that $\mathfrak{C}_1 = \mathcal{C}$.

Proposition 5.6. $\mathfrak{C}_1 \neq \mathfrak{C}_1^+ \cup \mathfrak{C}_1^-$.

Proof. As we saw above, by [Boh02, Corollary 2], any strongly quasipositive knot K has $\kappa_+(K) > 0$ and therefore $K \notin \mathfrak{C}_1^-$. However, Rudolph showed in [Rud83, Rud05] that one can construct a strongly quasipositive knot with any given Seifert pairing. As a result, we can find a strongly quasipositive knot K with positive Levine–Tristram signature. By Proposition 1.2 of [CHH13], $K \notin \mathcal{P}_0$ and therefore, $K \notin \mathfrak{C}_1^+$. \square

Clearly, any of the knots guaranteed by the above proposition must have both $\kappa_+(K)$ and $\kappa_-(K)$ non-zero and in fact, this condition characterizes all knots in $\mathfrak{C}_1 - (\mathfrak{C}_1^+ \cup \mathfrak{C}_1^-)$.

Proposition 5.7. $\mathfrak{C}_2 \neq \mathfrak{C}_1$, $\mathfrak{C}_2^+ \neq \mathfrak{C}_1^+$ and $\mathfrak{C}_2^- \neq \mathfrak{C}_1^-$.

Proof. The figure eight knot 4_1 is contained in both \mathfrak{C}_1^+ and \mathfrak{C}_1^- since it can be unknotted by changing a single positive or negative crossing. However, we know that $\text{Arf}(4_1) \neq 0$ and so by Corollary 1, it cannot bound a Casson tower of height two. Therefore, $4_1 \notin \mathfrak{C}_2$. Since $\mathfrak{C}_2^\pm \subseteq \mathfrak{C}_2$ the result follows.

Of course, any knot with $\text{Arf}(K) = 1$ lies in $\mathfrak{C}_1 - \mathfrak{C}_2$ by Corollary 1, since $\mathfrak{C}_1 = \mathcal{C}$. Similarly, any knot K with $\text{Arf}(K) = 1$ and unknotting number one lies in either $\mathfrak{C}_1^+ - \mathfrak{C}_2^+$ or $\mathfrak{C}_1^- - \mathfrak{C}_2^-$. \square

The above result shows that while the figure eight knot bounds a kinky disk with a single positive (resp. negative) kink, it cannot be extended to a Casson tower of height two. In fact, by Corollary 1, the figure eight knot does not bound *any* height two Casson tower, regardless of the number (and sign) of kinks at the first stage.

Corollary 5.8. $\mathfrak{C}_{2,0}^+ \equiv \mathfrak{C}_2^+ \neq \mathcal{P}_0$. Similarly, $\mathfrak{C}_{2,0}^- \equiv \mathfrak{C}_2^- \neq \mathcal{N}_0$.

Proof. This follows immediately from the previous proposition since $\mathfrak{C}_1^+ \subseteq \mathcal{P}_0$ and $\mathfrak{C}_1^- \subseteq \mathcal{N}_0$. \square

Recall that T_n^\pm denotes the twist knot with n twists, where the superscript denotes the sign of the clasp (see Figure 5.1).

Proposition 5.9. For even n , $T_n^+ \in \mathfrak{C}_{2,0}^+$ and $T_n^- \in \mathfrak{C}_{2,0}^-$.

Notice that by Corollary 1, knots in $\mathfrak{C}_{2,0}^\pm$ must have zero Arf invariant. As a result, for odd n , T_n^\pm cannot be contained in $\mathfrak{C}_{2,0}^\pm$, since $\text{Arf}(T_n^\pm) \equiv n \pmod{2}$. **highlight examples more**

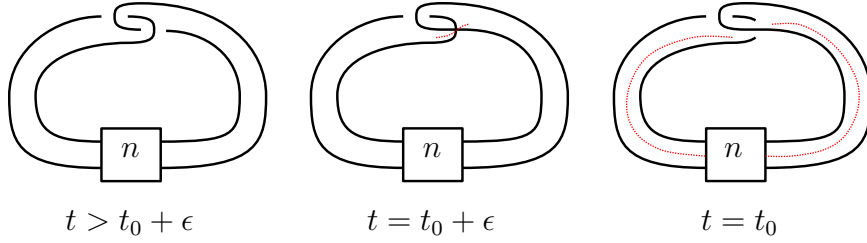
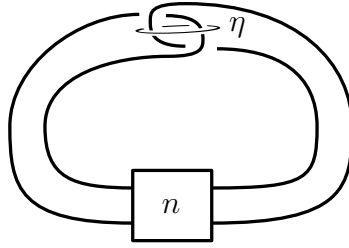


Figure 5.3: Homotopy showing the base-level kinky disk bounded by T_n^\pm . α , the standard curve to which the second-level kinky disk should be attached, is shown dotted.

Proof of Proposition 5.9. Let K denote T_{2k}^\pm for some $k \in \mathbb{Z}$. K bounds an obvious kinky disk D_1 in B^4 with a single positive (resp. negative) kink, corresponding to changing one of the two crossings at the clasp. The standard curve, which would need to bound a second stage kinky disk, is an unknot which can be seen as the ‘core’ curve of K , shown in Figure 5.3. Call this curve α . As depicted in the figure, α is ‘mostly’ contained in a single slice of B^4 (with respect to the radial function). Let this radius be denoted t_0 . D_1 is contained in the region of B^4 with radii $\geq t_0$, and as a result, we see that α bounds an *embedded* disk \widetilde{D}_2 away from D_1 , on the side of B^4 with radius $< t_0$. However, a regular neighborhood of \widetilde{D}_2 does not have the correct framing—it is twisted $2k$ times.

Around the (single) kink in D_1 , we have a linking torus T , which intersects \widetilde{D}_2 transversely once. For a precise description of the linking torus at the transverse point of intersection of two planes, see [FQ90, p. 12]. All we will need here is that the meridian and longitude of the linking torus are respectively meridians of the intersecting planes. Therefore, in our case, they are both meridians of D_1 .

Assume T is oriented such that $T \cdot \widetilde{D}_2 = -1$. Take k parallel (non-intersecting) copies of T . We can smooth the intersection between each copy of T and \widetilde{D}_2 to obtain a connected surface bounded by α . The embedded surface Σ thus obtained is homologically $\widetilde{D}_2 + kT$. The smoothing process is described in [GS99, p. 38] and can be performed without introducing any self-intersections of Σ .

Figure 5.4: The curve η .

The framing of Σ is the homological self-intersection number and changes by $2\widetilde{D}_2 \cdot kT = -2k$. We now have a correctly framed surface of genus k bounded by α . We will now use surgery to obtain a kinky disk bounded by α .

Assume $k = 1$ for the moment. Then Σ is a genus one surface. Consider the $(1, -1)$ curve on Σ . The meridian and longitude of Σ are the same as the meridian and longitude of T , and therefore the $(1, -1)$ curve on Σ is isotopic to the curve η shown in Figure 5.4, in the exterior of D_1 . For larger values of k we can find a set of curves, shown in Figure 5.5, which are each isotopic to $\eta \subseteq S^3$ away from D_1 . These curves are the images of the $(1, -1)$ curves on T in Σ —this is easily seen from the construction of Σ . Surgering along these curves (away from D_1) would give us a (correctly framed) second stage kinky handle and complete the proof. The resulting disk will have the correct framing since surgery does not change framing.

η bounds a genus one surface away from D_1 as shown in Figure 5.6. The longitude of this surface is isotopic (away from D_1) to the standard curve of D_1 , namely α . We know that α bounds a disk, \widetilde{D}_2 , away from D_1 . Surgering using parallel copies of \widetilde{D}_2 , we see that η bounds an immersed disk δ away from D_1 . Note that δ will necessarily

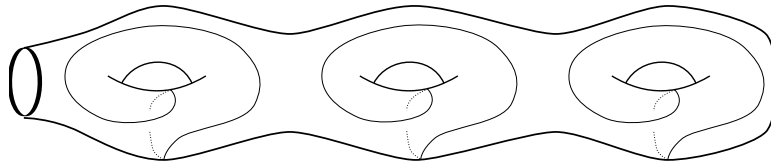


Figure 5.5:

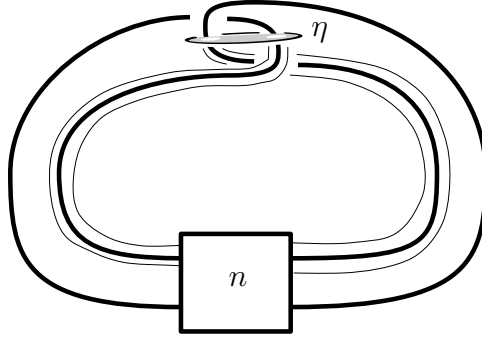


Figure 5.6: The curve η bounds a surface.

intersect \widetilde{D}_2 (and therefore Σ).

We can use δ to surger Σ when $k = 1$. For larger values of k , we will need multiple parallel copies of δ , which will necessarily intersect one another. However, as long as there are no intersections with D_1 , we still create a Casson tower of height two as desired¹. \square

Recall that $\text{Wh}_n^\pm(K)$ denotes the n -twisted Whitehead double of the knot K , where the superscript indicates the type of clasp. By a very similar argument as above, we can show the following.

Proposition 5.10. *For even n and any knot K , $\text{Wh}_n^+(K) \in \mathfrak{C}_{2,0}^+$ and $\text{Wh}_n^-(K) \in \mathfrak{C}_{2,0}^-$.*

Proof. The argument in this case differs from the proof of the previous proposition only in a few details. As before, $\text{Wh}_{2k}^\pm(K)$ bounds a first stage kinky disk D_1 with a single positive (resp. negative) kink. The standard curve α is no longer an unknot as in the previous case, but has the knot type of K . However, any knot K bounds a (correctly framed) kinky disk in the 4-ball, since it can be unknotted by changing crossings (see Example 5.1). However, the n twists in the Whitehead doubling operator used imply that a regular neighborhood of the naïve choice of second stage disk, \widetilde{D}_2 , is twisted $2k$ times. Fortunately, as before, we can tube with the linking torus at

¹The author is grateful to Robert Gompf for suggesting a key step in the proof for Proposition 5.9.

the (single) kink in the first stage disk, and surger repeatedly using copies of \widetilde{D}_2 to obtain a Casson tower of height two. The proof is identical to the proof of Proposition 5.9 apart from the fact that \widetilde{D}_2 is no longer embedded. Several new intersections are created as before, but they are all in the second stage kinky disk. \square

Corollary 5.11. $\mathfrak{C}_{2,0}^+ \neq \mathfrak{C}_{2,1}^+$, $\mathfrak{C}_{2,0}^- \neq \mathfrak{C}_{2,1}^-$, and $\mathfrak{C}_{2,0} \neq \mathfrak{C}_{2,1}$.

Proof. The knots T_n^+ and $\text{Wh}_n^+(K)$ are algebraically slice exactly when $n = l(l+1)$ with $l \geq 0$ [CG86]. (Similarly, knots T_n^- and $\text{Wh}_n^-(K)$ are algebraically slice exactly when $n = -l(l+1)$ with $l \geq 0$.) This fact, together with Proposition 5.9, yields infinitely many knots in $\mathfrak{C}_{2,0}^+ - \mathfrak{C}_{2,1}^+$, $\mathfrak{C}_{2,0}^- - \mathfrak{C}_{2,1}^-$, and $\mathfrak{C}_{2,0} - \mathfrak{C}_{2,1}$. This is because, by Corollary 2, knots in $\mathfrak{C}_{2,1}^\pm$ or $\mathfrak{C}_{2,1}$ must be algebraically slice. \square

Corollary 5.12. $\mathfrak{C}_3^+ \neq \mathfrak{C}_2^+$, $\mathfrak{C}_3^- \neq \mathfrak{C}_2^-$, and $\mathfrak{C}_3 \neq \mathfrak{C}_2$.

Proof. Since $\mathfrak{C}_3^\pm \subseteq \mathfrak{C}_{2,1}^\pm$ and $\mathfrak{C}_3 \subseteq \mathfrak{C}_{2,1}$, this follows immediately from the previous corollary. \square

From the proof of Proposition 5.10, it is tempting to speculate that iterated twisted Whitehead doubles bound arbitrarily high Casson towers, i.e. if a knot K bounds a Casson tower of height p , $\text{Wh}_n^\pm(K)$ bounds a Casson tower of height $p+1$ for any n . Unfortunately, this does not follow when $n \neq 0$. In particular, if our wishful thinking were correct, twist knots would bound arbitrarily high Casson towers (since they are twisted doubles of the unknot, which bounds arbitrarily high Casson towers). However, we know this is not true since some twist knots are not algebraically slice and therefore, do not bound Casson towers of height three.

In the $n = 0$ case, we get the following result.

Proposition 5.13. *For any knot $K \in \mathfrak{C}_k$,*

$$\text{Wh}_0^+(K) \in \mathfrak{C}_{k+1}^+$$

and

$$Wh_0^-(K) \in \mathfrak{C}_{k+1}^-.$$

Proof. If a knot J bounds a Casson tower of height k , $Wh_0^\pm(J)$ bounds a Casson tower of height $k + 1$, with a single kink in the lowest level with sign corresponding to the sign of the clasp of the pattern used. The result follows. \square

Remark 5.14. Note that the above proposition implies that for any knot $K \in \mathfrak{C}_k$,

$$Wh_0^+(\underbrace{Wh_0^\pm \cdots Wh_0^\pm(K)}_{n-1 \text{ times}}) \in \mathfrak{C}_{n+k}^+$$

and

$$Wh_0^-(\underbrace{Wh_0^\pm \cdots Wh_0^\pm(K)}_{n-1 \text{ times}}) \in \mathfrak{C}_{n+k}^-.$$

The following is an immediate corollary of the above proposition, Theorem A, Proposition 5.10 and Corollary 3.8.

Corollary 5.15. *For any even k and any knot K ,*

$$Wh_0^+(Wh_k^\pm(K)) \in \mathfrak{C}_3^+ \subseteq \bigcap_n \mathfrak{C}_{2,n}^+ \subseteq \bigcap_n \mathcal{P}_n$$

and

$$Wh_0^-(Wh_k^\pm(K)) \in \mathfrak{C}_3^- \subseteq \bigcap_n \mathfrak{C}_{2,n}^- \subseteq \bigcap_n \mathcal{N}_n.$$

The above is related to Corollary 3.7 in [CHH13], which shows that if $J \in \mathcal{P}_0$ then $Wh_0^\pm(J)$ is in $\bigcap_n \mathcal{P}_n$. For any K and n , we know that $Wh_n^+(K) \in \mathcal{P}_0$, and therefore it was already known that $Wh_0^+(Wh_n^+(K)) \in \bigcap_n \mathcal{P}_n$. However, it is **not** generally true that $Wh_n^-(K) \in \mathcal{P}_0$ for any K . For example, $Wh_0^-(LHT) \notin \mathcal{P}_0$.

Generalization to links

The definitions of \mathfrak{C}_n , \mathfrak{C}_n^\pm , $\mathfrak{C}_{2,n}$ and $\mathfrak{C}_{2,n}^\pm$ can be naturally generalized to the context of links. Since the connected sum operation is not well-defined on links, we have to consider the *string link concordance group* of m -component string links, denoted $\mathcal{C}(m)$, under the concatenation operation. For $L \in \mathcal{C}(m)$, let \widehat{L} denote the m -component link obtained by taking the closure of L .

Definition 1'. An m -component string link L is said to be in $\mathfrak{C}_n(m)$ if \widehat{L}_i , the components of \widehat{L} , bound disjoint Casson towers of height n .

Definition 2'. An m -component string link L is said to be in $\mathfrak{C}_{2,n}(m)$ if \widehat{L}_i , the components of \widehat{L} , bound disjoint Casson towers T_i of height two such that each member of a standard set of curves for each T_i is in $\pi_1(B^4 - \sqcup_i T_i)^{(n)}$.

Definition 3'. An m -component string link L is said to be in $\mathfrak{C}_n^+(m)$ (resp. $\mathfrak{C}_n^-(m)$) if \widehat{L}_i , the components of \widehat{L} , bound disjoint Casson towers of height n such that all the kinks in the first stage kinky disks are positive (resp. negative).

Definition 4'. An m -component string link L is said to be in $\mathfrak{C}_{2,n}^+(m)$ (resp. $\mathfrak{C}_{2,n}^-(m)$) if \widehat{L}_i , the components of \widehat{L} , bound disjoint Casson towers T_i of height two such that all

the kinks in the first stage kinky disks are positive (resp. negative) and each member of a standard set of curves for each T_i is in $\pi_1(B^4 - \sqcup_i T_i)^{(n)}$.

There are similar definitions for the grope filtrations $\mathcal{G}_n(m)$ and $\mathcal{G}_{2,n}(m)$, and the n -solvable filtration $\mathcal{F}_n(m)$ for $\mathcal{C}(m)$ which we omit for the sake of brevity—they are identical to the definitions in the case of knots, except that the components of the link are required to bound *disjoint* disks in 4-manifolds of the relevant flavor. Positive links, i.e. links in $\mathcal{P}_0(m)$, have been studied by Cochran–Tweedy in [CT].

Since all of our arguments in Chapter 3 take place within Casson towers, the results generalize easily to links. Therefore, we obtain the following theorem.

Theorem A'. *For any $n \geq 0$, and $m \geq 1$,*

$$(i) \ \mathfrak{C}_{n+2}(m) \subseteq \mathcal{G}_{n+2}(m) \subseteq \mathcal{F}_n(m)$$

$$(ii) \ \mathfrak{C}_{2,n}(m) \subseteq \mathcal{G}_{2,n}(m) \subseteq \mathcal{F}_n(m).$$

$$(iii) \ \mathfrak{C}_{n+2}^+(m) \subseteq \mathfrak{C}_{2,n}^+(m) \subseteq \mathcal{P}_n(m)$$

$$(iv) \ \mathfrak{C}_{n+2}^-(m) \subseteq \mathfrak{C}_{2,n}^-(m) \subseteq \mathcal{N}_n(m)$$

Note that $\mathfrak{C}_1^+(m) \subseteq \mathcal{P}_0(m)$ and $\mathfrak{C}_1^-(m) \subseteq \mathcal{N}_0(m)$, even in the case of links. Using a near-identical proof to that of Theorem B, we obtain the following.

Theorem B'. *For any m -component string link L , the following statements are equivalent.*

$$(a) \ L \in \mathfrak{C}_1^+(m) \text{ (resp. } \mathfrak{C}_1^-(m)\text{)}$$

(b) \widehat{L} is concordant to a link each of whose components is a fusion knot of a split collection of positive (resp. negative) Hopf links

(c) \widehat{L} is concordant to a link each of whose components can be changed to a ribbon knot by changing only positive (resp. negative) crossings (within the same component).

Recall that any knot K lies in \mathfrak{C}_1 since it can be unknotted by changing some number of crossings. However, it is not true that every m -component link lies in $\mathfrak{C}_1(m)$ as we see below.

Proposition 6.1. *If an m -component string link L lies in $\mathfrak{C}_1(m)$, then \widehat{L} is link homotopic to the m -component unlink and the pairwise linking numbers of \widehat{L} are zero.*

Proof. Since $L \in \mathfrak{C}_1(m)$, the components of \widehat{L} bound disjoint immersed disks in B^4 . By following the proof of Theorem B', we see that \widehat{L} is concordant to a link \widehat{M} which can be changed to a ribbon link by changing some number of crossings, i.e. \widehat{M} is link homotopic to a ribbon link. However, we know from [Gif79, Gol79] that link concordance implies link homotopy. Since \widehat{L} is concordant to \widehat{M} and any m -component ribbon link is concordant to the m -component unlink, we have that \widehat{L} is link homotopic to \widehat{M} which is link homotopic to a ribbon link which is link homotopic to the m -component unlink.

The linking number between two simple closed curves in S^3 can be computed as the signed intersection number between 2-chains bounded by them in B^4 [Rol90, p. 136]. Since the components of \widehat{L} bound disjoint 2-chains (in particular, immersed disks) in B^4 all the pairwise linking numbers are zero. \square

As in Corollary 3.3, we obtain the following result.

Corollary 3.3'. *Let $\mathcal{T}(m)$ denote the set of all topologically slice string links with m*

components. Then, for any $m \geq 1$,

$$\mathcal{T}(m) \subseteq \bigcap_{n=1}^{\infty} \mathcal{G}_n(m).$$

As we mentioned in Chapter 2, the groups $\mathcal{G}_{2,n}(m)$ have not appeared previously in the literature, but several results relating to the grope filtration carry over easily. In the case of links, this can be seen in context of k -cobordism of links ([Coc90, Definition 9.1][Sat84]) as follows. We reference the corresponding results from [Ott12] regarding the grope filtration below since our proofs are essentially the same.

Proposition 6.2 (Proposition 6.4 of [Ott12]). *If $L \in \mathcal{G}_{2,n}(m)$ then L is 2^{n+1} -cobordant to a slice link, i.e. L is null 2^{n+1} -cobordant.*

Proof. The proof is essentially identical to the proof of Proposition 6.4 of [Ott12], which says that if $L \in \mathcal{G}_{n+2}(m)$ then L is null 2^{n+1} -cobordant. Her proof only uses the fact that each member of a symplectic basis for the first stage surfaces (call them Σ_i) of the gropes lies in $\pi_1(B^4 - \sqcup_i \Sigma_i)$, which clearly still holds for a link in $\mathcal{G}_{2,n}(m)$. \square

Corollary 2.2 of [Lin91] states that if a link L is null k -cobordant, then Milnor's $\bar{\mu}$ -invariants of L with length less than or equal to $2k$ vanish. Therefore, we obtain the following corollary.

Corollary 6.3 (Corollary 6.6 of [Ott12]). *If $L \in \mathcal{G}_{2,n}(m)$, then $\bar{\mu}_L(I) = 0$ for $|I| \leq 2^{n+2}$.*

Since $\mathfrak{C}_{2,n}(m) \subseteq \mathcal{G}_{2,n}(m)$ for all n and m , we also obtain the following.

Corollary 6.4 (Corollary 6.6 of [Ott12]). *If $L \in \mathcal{C}_{2,n}(m)$, then $\bar{\mu}_L(I) = 0$ for $|I| \leq 2^{n+2}$.*

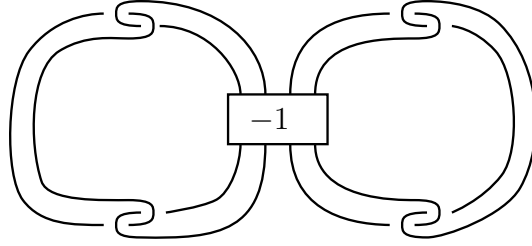


Figure 6.1: The above link is in $\mathcal{P}_0(4)$ but not in either $\mathfrak{C}_1^\pm(4)$. The strands going through the box marked with -1 are given a full negative twist relative to one another.

Proposition 6.5. For $m \geq 2^{n+2}$ and $n \geq 0$,

- (a) $\mathbb{Z} \subseteq \mathcal{F}_n(m)/\mathcal{G}_{2,n}(m)$,
- (b) $\mathbb{Z} \subseteq \mathcal{P}_n(m)/\mathcal{G}_{2,n}(m)$,
- (c) $\mathbb{Z} \subseteq \mathcal{N}_n(m)/\mathcal{G}_{2,n}(m)$.

Proof. The proof of (a) is very closely related to Otto's proof of [Ott12, Corollary 6.8] in light of Corollary 6.3. Here is a short sketch. Let H denote the positive Hopf link, and $BD^i(H)$ its i^{th} iterated Bing double (where each component of a link gets Bing doubled at each step). Otto shows that $BD^{n+1}(H) \in \mathcal{F}_n$ for each n . Work of Cochran [Coc90, Theorem 8.1] then shows that $\bar{\mu}_{BD^{n+1}(H)}(I) = 1$ for some I of length 2^{n+2} with distinct indices (note that $BD^{n+1}(L)$ has 2^{n+2} components) and additionally, all $\bar{\mu}$ -invariants of smaller length vanish. Corollary 6.3 shows that $BD^{n+1}(H) \in \mathcal{F}_n/\mathcal{G}_{2,n}$ for each n . Since the first non-zero $\bar{\mu}$ -invariant is additive under concatenation of string links [Coc90, Theorem 8.13][Orr89], we see that $BD^{n+1}(H)$ generates an infinite cyclic subgroup of $\mathcal{F}_n(m)/\mathcal{G}_{2,n}(m)$.

We give the proof for (b); taking concordance inverses of these examples will complete the proof for (c). Consider the link L given in Figure 6.1. We see that $L \in \mathcal{P}_0(m)$ in [CT], since it is obtained from an unlink by adding a *generalized positive crossing*. By [CP12, Lemma 3.7], $BD^n(L) \in \mathcal{P}_n(m)$. However, $BD^n(L)$ is

link homotopic to $BD^{n+1}(H)$. Since $\bar{\mu}$ -invariants with distinct indices are invariants of link homotopy, we can complete the proof as in part (a). \square

In the case of links we also obtain the following additional results, which we are currently unable to prove in the case of knots.

Proposition 6.6. $\mathcal{P}_0(m) \neq \mathfrak{C}_1^+(m)$ and $\mathcal{N}_0(m) \neq \mathfrak{C}_1^-(m)$ for $m \geq 4$.

Proof. Links demonstrating this inequality may be found in [CT, Example 4.13]. The link L shown in Figure 6.1 is link homotopic to the Bing double of a Hopf link (this is easily seen by drawing a picture of both links; recall that the box in Figure 6.1 containing a ‘1’ indicates a full twist of all the strands passing through it) and therefore, has non-zero $\bar{\mu}(1234)$. This implies that L is not link homotopic to the unlink, and therefore, by Proposition 6.1 is not in $\mathfrak{C}_1^+(m)$. However, we see that $L \in \mathcal{P}_0(m)$ in [CT].

The mirror image of the link in Figure 6.1 is in $\mathcal{N}_0(m) - \mathfrak{C}_1^-(m)$. \square

We can actually do better by following the proof of Proposition 6.5, as follows.

Proposition 6.7. For $m \geq 2^{n+2}$ and $n \geq 0$,

$$(a) \mathbb{Z} \subseteq \mathcal{F}_n(m) / \mathfrak{C}_1(m),$$

$$(b) \mathbb{Z} \subseteq \mathcal{P}_n(m) / \mathfrak{C}_1^+(m),$$

$$(c) \mathbb{Z} \subseteq \mathcal{N}_n(m) / \mathfrak{C}_1^-(m).$$

Proof. In the proof of Proposition 6.5, we demonstrated the existence of links which are in $\mathcal{F}_n(m)$ (resp. $\mathcal{P}_n(m)$, $\mathcal{N}_n(m)$) and have a non-zero $\bar{\mu}$ -invariant with distinct indices. These links are therefore not link homotopic to the unlink, and as a result, by Proposition 6.1 are not contained in $\mathfrak{C}_1(m)$ (resp. $\mathfrak{C}_1^+(m)$, $\mathfrak{C}_1^-(m)$). \square

References

- [Ada04] Colin C. Adams, *The knot book*, American Mathematical Society, Providence, RI, 2004, An elementary introduction to the mathematical theory of knots, Revised reprint of the 1994 original. MR 2079925 (2005b:57009) 1.1.1
- [Baa05] Sebastian Baader, *Slice and Gordian numbers of track knots*, Osaka J. Math. **42** (2005), no. 1, 257–271. MR 2132015 (2005m:57005) 4.1
- [Boh02] Christian Bohr, *On the kinkiness of closed braids*, J. Knot Theory Ramifications **11** (2002), no. 7, 1155–1163. MR 1941690 (2003k:57006) 5.2, 5, 5
- [Cas86] Andrew J. Casson, *Three lectures on new-infinite constructions in 4-dimensional manifolds*, À la recherche de la topologie perdue, Progr. Math., vol. 62, Birkhäuser Boston, Boston, MA, 1986, With an appendix by L. Siebenmann, pp. 201–244. MR 900253 1.1.2, 2.1
- [CG86] A. J. Casson and C. McA. Gordon, *Cobordism of classical knots*, À la recherche de la topologie perdue, Progr. Math., vol. 62, Birkhäuser Boston, Boston, MA, 1986, With an appendix by P. M. Gilmer, pp. 181–199. MR 900252 5
- [CG88] Tim D. Cochran and Robert E. Gompf, *Applications of Donaldson’s theorems to classical knot concordance, homology 3-spheres and property P*, Topology **27** (1988), no. 4, 495–512. MR 976591 (90g:57020) 1.2, 4.1
- [CH12] Tim D. Cochran and Peter D. Horn, *Structure in the bipolar filtration of topologically slice knots*, Preprint: <http://arxiv.org/abs/1208.5788>, 2012. 1.1.1
- [Cha14] Jae Choon Cha, *Symmetric Whitney tower cobordism for bordered 3-manifolds and links*, Trans. Amer. Math. Soc. **366** (2014), no. 6, 3241–3273. MR 3180746 3

-
- [CHH13] Tim D. Cochran, Shelly Harvey, and Peter Horn, *Filtering smooth concordance classes of topologically slice knots*, *Geom. Topol.* **17** (2013), no. 4, 2103–2162. MR 3109864 1.1.1, 1.2, 2.1, 2.3, 4.1, 5.2, 5.4, 5, 5
- [CL86] Tim D. Cochran and W. B. Raymond Lickorish, *Unknotting information from 4-manifolds*, *Trans. Amer. Math. Soc.* **297** (1986), no. 1, 125–142. MR 849471 (87i:57003) 1.2, 4.1, 5.2
- [Coc90] Tim D. Cochran, *Derivatives of links: Milnor’s concordance invariants and Massey’s products*, *Mem. Amer. Math. Soc.* **84** (1990), no. 427, x+73. MR 1042041 (91c:57005) 6, 6
- [COT03] Tim D. Cochran, Kent E. Orr, and Peter Teichner, *Knot concordance, Whitney towers and L^2 -signatures*, *Ann. of Math. (2)* **157** (2003), no. 2, 433–519. MR 1973052 (2004i:57003) 1.1.1, 1.2, 2.2, 2.3, 2.6, 2.10, 2.2
- [CP12] Jae Choon Cha and Mark Powell, *Covering link calculus and the bipolar filtration of topologically slice links*, Preprint: <http://arxiv.org/abs/1212.5011>, to appear: *Geometry and Topology*, 2012. 1.1.1, 6
- [CT] Tim D. Cochran and Eamonn Tweedy, *Positive links*, Preprint: <http://arxiv.org/abs/1303.6691>, to appear: *Algebraic & Geometric Topology*. 4.1, 6, 6, 6
- [FQ90] Michael H. Freedman and Frank Quinn, *Topology of 4-manifolds*, Princeton Mathematical Series, vol. 39, Princeton University Press, Princeton, NJ, 1990. MR 1201584 (94b:57021) 5
- [Fre82] Michael H. Freedman, *The topology of four-dimensional manifolds*, *J. Differential Geom.* **17** (1982), no. 3, 357–453. MR 679066 (84b:57006) 1.1.2, 1.1.2, 1.2, 2.1, 2.1
- [Fre88] ———, *Whitehead₃ is a “slice” link*, *Invent. Math.* **94** (1988), no. 1, 175–182. MR 958596 (89h:57004) 2.1
- [Gif79] Charles H. Giffen, *Link concordance implies link homotopy*, *Math. Scand.* **45** (1979), no. 2, 243–254. MR 580602 (82a:57005) 6
- [Gol79] Deborah L. Goldsmith, *Concordance implies homotopy for classical links in M^3* , *Comment. Math. Helv.* **54** (1979), no. 3, 347–355. MR 543335 (80h:57006) 6
- [Gom84] Robert E. Gompf, *Infinite families of Casson handles and topological disks*, *Topology* **23** (1984), no. 4, 395–400. MR 780732 (86i:57040) 1.1.2
- [Gom86] ———, *Smooth concordance of topologically slice knots*, *Topology* **25** (1986), no. 3, 353–373. MR 842430 (87i:57004) 1.1.1, 1.2, 4.1, 5

-
- [Gom89] ———, *Periodic ends and knot concordance*, Proceedings of the 1987 Georgia Topology Conference (Athens, GA, 1987), vol. 32, 1989, pp. 141–148. MR 1007986 (90f:57004) 1.1.2
- [Gom05] ———, *Stein surfaces as open subsets of \mathbb{C}^2* , J. Symplectic Geom. **3** (2005), no. 4, 565–587, Conference on Symplectic Topology. MR 2235855 (2007f:32031) 1.2, 2.1, 3
- [GS84] Robert E. Gompf and Sukhjit Singh, *On Freedman’s reimbedding theorems*, Four-manifold theory (Durham, N.H., 1982), Contemp. Math., vol. 35, Amer. Math. Soc., Providence, RI, 1984, pp. 277–309. MR 780584 (86i:57018) 1.1.2, 1.2, 2.1, 2.1
- [GS99] Robert E. Gompf and András I. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, vol. 20, American Mathematical Society, Providence, RI, 1999. MR 1707327 (2000h:57038) 1.1.1, 2.1, 5
- [Hed10] Matthew Hedden, *Notions of positivity and the Ozsváth-Szabó concordance invariant*, J. Knot Theory Ramifications **19** (2010), no. 5, 617–629. MR 2646650 (2011j:57020) 1
- [Hor10] Peter D. Horn, *The non-triviality of the grope filtrations of the knot and link concordance groups*, Comment. Math. Helv. **85** (2010), no. 4, 751–773. MR 2718138 (2012b:57016) 2.7
- [Kir89] Robion C. Kirby, *The topology of 4-manifolds*, Lecture Notes in Mathematics, vol. 1374, Springer-Verlag, Berlin, 1989. MR 1001966 (90j:57012) 2.1
- [Lic62] W. B. R. Lickorish, *A representation of orientable combinatorial 3-manifolds*, Ann. of Math. (2) **76** (1962), 531–540. MR 0151948 (27 #1929) 1.1
- [Lin91] Xiao-Song Lin, *Null k -cobordant links in S^3* , Comment. Math. Helv. **66** (1991), no. 3, 333–339. MR 1120650 (93e:57034) 6
- [Liv04] Charles Livingston, *Computations of the Ozsváth-Szabó knot concordance invariant*, Geom. Topol. **8** (2004), 735–742 (electronic). MR 2057779 (2005d:57019) 4.1
- [Orr89] Kent E. Orr, *Homotopy invariants of links*, Invent. Math. **95** (1989), no. 2, 379–394. MR 974908 (90a:57012) 6
- [OS03] Peter Ozsváth and Zoltán Szabó, *Knot Floer homology and the four-ball genus*, Geom. Topol. **7** (2003), 615–639. MR 2026543 (2004i:57036) 4.1

-
- [Ott12] Carolyn Otto, *The (n) -solvable filtration of the link concordance group and milnor's invariants*, Preprint: <http://arxiv.org/abs/1211.1423>, to appear: Algebraic & Geometric Topology, 2012. 6, 6.2, 6, 6.3, 6.4, 6
- [Prz89] Józef H. Przytycki, *Positive knots have negative signature*, Bull. Polish Acad. Sci. Math. **37** (1989), no. 7-12, 559–562 (1990). MR 1101920 (92a:57010) 4.1
- [Qui82] Frank Quinn, *Ends of maps. III. Dimensions 4 and 5*, J. Differential Geom. **17** (1982), no. 3, 503–521. MR 679069 (84j:57012) 1.2, 2.1, 3
- [Ras03] Jacob A. Rasmussen, *Floer homology and knot complements*, Ph.D. thesis, Harvard University, 2003. 4.1
- [Rol90] Dale Rolfsen, *Knots and links*, Mathematics Lecture Series, vol. 7, Publish or Perish Inc., Houston, TX, 1990, Corrected reprint of the 1976 original. MR 1277811 (95c:57018) 6
- [Rud83] Lee Rudolph, *Constructions of quasipositive knots and links. I*, Knots, braids and singularities (Plans-sur-Bex, 1982), Monogr. Enseign. Math., vol. 31, Enseignement Math., Geneva, 1983, pp. 233–245. MR 728589 (86k:57004) 4.1, 5
- [Rud89] ———, *Quasipositivity and new knot invariants*, Rev. Mat. Univ. Complut. Madrid **2** (1989), no. 1, 85–109. MR 1012106 (90k:57009) 4.1
- [Rud99] ———, *Positive links are strongly quasipositive*, Proceedings of the Kirbyfest (Berkeley, CA, 1998), Geom. Topol. Monogr., vol. 2, Geom. Topol. Publ., Coventry, 1999, pp. 555–562 (electronic). MR 1734423 (2000j:57015) 4.1, 5
- [Rud05] ———, *Knot theory of complex plane curves*, Handbook of knot theory, Elsevier B. V., Amsterdam, 2005, pp. 349–427. MR 2179266 (2007b:57016) 4.1, 4.1, 5
- [Sat84] Nobuyuki Sato, *Cobordisms of semiboundary links*, Topology Appl. **18** (1984), no. 2-3, 225–234. MR 769293 (86d:57010) 6
- [Sch06] Rob Schneiderman, *Whitney towers and gropes in 4-manifolds*, Trans. Amer. Math. Soc. **358** (2006), no. 10, 4251–4278 (electronic). MR 2231378 (2007e:57018) 1.2
- [Siv13] Steven Sivek, *Is every quasipositive knot strongly quasipositive?*, <http://mathoverflow.net/questions/132746/is-every-quasipositive-knot-strongly-quasipositive>, 2013. 1
- [Tra88] Paweł Traczyk, *Nontrivial negative links have positive signature*, Manuscripta Math. **61** (1988), no. 3, 279–284. MR 949818 (89g:57010) 4.1

- [Wal60] Andrew H. Wallace, *Modifications and cobounding manifolds*, *Canad. J. Math.* **12** (1960), 503–528. MR 0125588 (23 #A2887) 1.1