TOPOLOGICAL 4-MANIFOLDS: THE DISC EMBEDDING THEOREM AND BEYOND TECH TOPOLOGY SUMMER SCHOOL

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Abstract. The goal of this mini-course is to introduce the work of Freedman and Quinn on topological 4-manifolds, and describe the main open problems in the area. The first four lectures contain an outline of a proof of the topological Poincaré conjecture in dimension four, due to Freedman: every topological homotopy 4-sphere is homeomorphic to S^4. The final lecture is a survey of other work and open problems.

Contents

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1. Outline of the course

The field of 4-manifold topology was revolutionised in the 1980s by concurrent work of Freedman and Donaldson, which established a vast disparity between the behaviour of smooth vs topological 4-manifolds. Soon after, Quinn expanded on Freedman's techniques and established fundamental tools for topological 4-manifolds, such as transversality. The goal of this mini-course is to introduce the work of Freedman and Quinn, focussing primarily on topological 4-manifolds, explaining the motivation and context for their work, illustrating the key tools and ideas, and outlining potential future directions. My hope is to convince you of the following claims:

- The topological category is not as scary as you might think.
- We know a lot about topological 4-manifolds, but there is much more to be done.
- Smooth manifold topologists should be interested in these topics.
- Smooth manifold topologists can also contribute to answers to the big open questions for topological 4-manifolds.

The results we will talk about in this mini-course have two flavours. Results of the first type, as also explained in the Stipsicz lectures (ttss.math.gatech.edu/stipsicz-mini-course), provide the context for several smooth results. For example, in order to find exotic smooth structures, we need on one hand to know when two 4-manifolds are homeomorphic (for which we need topological classification results) and on the other hand to know when they fail to be diffeomorphic (for which we need smooth invariants). Results of this form include

- the h- and s-cobordism theorems for good groups
- the exactness of the surgery sequence for good groups
- classification results for 4-manifolds up to homeomorphism

Results of the second type enable us to use tools and techniques familiar from smooth contexts in the topological setting. These are essential when working with topological 4-manifolds, and include

- the annulus theorem, which implies that connected sum of (oriented) topological 4-manifolds is well defined
- the immersion lemma and topological transversality, that continuous maps may be approximated by maps with transverse intersections
- the existence of normal bundles for locally flat submanifolds

We will address results of both types in the mini-course.

1.1. Recorded talks.

The recorded talks made available before the summer school will focus on the following theorem.

Theorem 1.1. *Let M and N be closed, smooth, oriented, simply connected 4-manifolds which have the same intersection form. Then they are homeomorphic.*

The proof has two steps. The first is a result of Wall, saying that such 4-manifolds are smoothly *h-cobordant*. The second is Freedman's result, that smooth, compact, simply connected h-cobordisms are topologically trivial. The proof of Freedman's result will use the *disc embedding theorem*, in order to find *Whitney discs* along which one may perform the *Whitney trick*. All of these notions will be explained.

Video titles.

- Lecture 1
  a. Course overview
  b. Proof of the high-dimensional h-cobordism theorem
  c. Proof of Wall's theorem and proof sketch for cork theorem
- Lecture 2
  a. Visualising surfaces for the Whitney trick in dimension four
  b. Proof of Freedman's h-cobordism theorem, modulo the disc embedding theorem
  c. Outline of proof of the disc embedding theorem

1.2. **Live talks.** The focus of the first two live talks will be the work of Quinn, establishing fundamental tools in topological 4-manifold topology. While the recorded lectures prove that every smooth homotopy 4-sphere is homeomorphic to S^4, the first two live talks will explain how to upgrade this result to be fully topological, i.e. that every topological homotopy 4-sphere is homeomorphic to S^4. A key technical result is *handle smoothing*, which we will use to show that noncompact, connected topological 4-manifolds are smoothable.

The final talk will briefly describe related topics which we did not explore in the previous talks, but focus on open problems and potential future directions.

1.3. **Conventions.**

- The symbol $\simeq$ denotes homotopy equivalence
- The symbol $\approx$ denotes homeomorphism
- The symbol $\cong$ denotes diffeomorphism

1.4. **Personnel, logistics, and resources.** The TAs for this course are:

- Patrick Orson (ETHZ/MPIM) people.math.ethz.ch/~porson
- Benjamin Ruppik (MPIM) ben300694.github.io

Aru is giving the lectures. Aru, Ben, Danica, Mark, and Patrick are typing up these lecture notes, sometimes live during the talks.

Videos of the lectures and further resources will be made available at ttss.math.gatech.edu/ray-mini-course. These notes will likely include details that were only outlined or outright skipped in the lectures.

There are many exercises in these lecture notes. **Green** exercises are usually straightforward and should be attempted if you are seeing this material for the first time. Prerequisites are courses in introductory geometric and algebraic topology. **Orange** exercises are for those who are already comfortable with some of the terminology; they may require nontrivial input from outside the mini-course(s). **Red** exercises are challenge problems.

2. h-cobordisms and Wall's theorem

As mentioned above, the proof of Theorem 1.1 uses the notion of h-cobordisms. We begin this section by defining h-cobordisms and proving Smale's high-dimensional h-cobordism theorem, to familiarise ourselves with the tools needed for the proof of Theorem 1.1.

Definition 2.1. Let M_0^n and M_1^n be smooth, compact, oriented n-manifolds. A smooth, compact, oriented $(n+1)$-manifold with $\partial W = -M_0^n \sqcup M_1^n$ is said to be an *h-cobordism* from M_0^n to M_1^n if the inclusion maps $\iota_i \colon M_i \to W$ are homotopy equivalences.

You should think of this as saying that, up to homotopy, h-cobordisms are products, i.e. of the form $M_0^n \times [0,1]$. The following is a fundamental result in high-dimensional topology.

Theorem 2.2 (Smale [Sma61, Sma62])**.** *Let $n \geq 5$, and W^{n+1} a smooth, compact, oriented, simply connected h-cobordism from M_0^n to M_1^n. Then $W \cong M_0 \times [0,1]$.*

More specifically, there exists a diffeomorphism $\varphi \colon W \to M_0 \times [0,1]$, where the restriction $\varphi|_{M_0} \colon M_0 \to M_0$ is the identity map. Note that the restriction $\varphi|_{M_1}$ is a diffeomorphism from M_1 to M_0.

As a straightforward corollary of the above, Smale proved the (category losing) high-dimensional Poincaré conjecture, for which he won the Fields medal.

Corollary 2.3. *Let $n \geq 6$. Every smooth homotopy n-sphere is homeomorphic to S^n.*

We now sketch a proof of Theorem 2.2. See also [Sco05, Chapter 1; Mil65; Sma60]. For more on the high-dimensional Poincaré conjectures, see [Sta60, Zee62, New66].

Proof of Theorem 2.2. First we note that, as a smooth, compact manifold, W admits a handle decomposition relative to M_0, i.e. there is an identification of W with the smooth manifold obtained by iteratively attaching finitely many handles to $M_0 \times [0,1]$ along $M_0 \times \{1\}$ via smooth handle attaching maps, followed by smoothing corners.

For more on the existence of handle decompositions, see [GS99, Sco05, Mil65, Mil63]. Briefly, we begin with a continuous map $W \to [0,1]$, approximate it by a smooth function, then in turn by a Morse function. Critical points of Morse functions correspond precisely to handles.

Remark 2.4. There are analogous notions of PL and topological handle decompositions, both in the absolute and relative settings, where handles are attached along PL and topological embeddings, respectively.

The main idea of the proof is to manipulate the handle decomposition of W until all the handles cancel out. A (smooth) handle decomposition relative to M_0 with no handles is, by definition, diffeomorphic to the product $M_0 \times [0,1]$. We will modify the handle decomposition by isotopies of the handle attaching maps, including handles slides, and handle cancellation (more on these moves in the Piccirillo lectures (ttss.math.gatech.edu/piccirillo-mini-course)). We will also need the following indispensable tool from differential topology.

Theorem 2.5 (Submanifold transversality in the smooth category)**.** *Given smooth submanifolds P^p and Q^q in an ambient manifold W^m, we may smoothly isotope P so that P and Q intersect transversely, i.e. the dimension of $P \pitchfork Q$ is $p+q-m$.*

In particular, if $p+q<m$, we may isotope P so that $P \cap Q = \emptyset$.

We now begin manipulating the handle decomposition of W relative to M_0.

Step 1. *Arrange that handles are attached in increasing order of index.*

It is relatively straightforward to see that if the handle h' is attached after the handle h, such that the attaching sphere of h' misses the belt sphere of h, then one may reorder the handle attachment so that h is attached after h'. This follows since the attaching sphere for h' can be isotoped away from all of h, for example, by transporting radially away from the belt sphere. Assume that h is a k-handle and h' is an l-handle. Then the dimension of the belt sphere of h is $n-k$ (recall that we are working with $(n+1)$-dimensional handles). The dimension of the attaching sphere for h' is $l-1$. The manifold after attaching h is n-dimensional. So, up to isotopy, we may assume that the intersection between the belt sphere of h and the attaching sphere of h' has dimension $(n-k)+(l-1)-n = l-k-1$. In particular, if $k \geq l$, the intersection can be assumed to be empty, and so we can reorder h and h'.

Step 2. *Cancel all* 0*-handles (using* 1*-handles).*

Recall that W is connected. Further, 0-handles are attached along their (empty) attaching region, and the only handles with nonempty, disconnected attaching region are index 1. Hence, at least one of the (finitely many) 0-handles must be attached to $M_0 \times \{1\}$ by a 1-handle, i.e. there is a 1-handle h_1 with one connected component of its attaching region in $M_0 \times \{1\}$ and the other in the belt sphere ($\cong S^n$) of the 0-handle h_0. In particular, the attaching sphere of h_1 intersects the belt sphere of h_0 precisely once, and the pair may be cancelled and removed from the handle decomposition. This process reduces the number of 0-handles in the handle decomposition by one, and by induction, we may assume that there are no 0-handles in the decomposition moving forward.

Remark 2.6. A similar argument shows that a handle decomposition for a closed n-manifold can be assumed to have a single 0-handle as well as, by turning the handle decomposition upside down, a single n-handle.

Step 1. *Trade* 1*-handles for* 3*-handles.*

Let $W_2 \subseteq W$ denote the union of $M_0 \times [0,1]$ and the 1- and 2-handles of W. Let M_2 denote the new boundary, so $\partial W_2 = -M_0 \sqcup M_2$.

Consider the chain of inclusion induced maps $\pi_1(M_0) \to \pi_1(W_2) \to \pi_1(W)$. Since W is built from W_2 by attaching handles of index strictly greater than 2, the second map is an isomorphism. The composition is an isomorphism by hypothesis. Thus the first map is an isomorphism.

Fix a 1-handle h_1 in W_2, with core arc α. We claim that there is an arc $\beta \subseteq M_0$ such that $\gamma := \alpha \cup \beta$ is a null-homotopic loop in W_2. To see this, first choose any arc β' with the same endpoints as α. Then there is some loop $\delta \subseteq M_0$ with the same image in $\pi_1(W_2)$ as $\alpha \cup \beta'$, since the inclusion induced map $\pi_1(M_0) \to \pi_2(W_2)$ is surjective. The connected sum of β' and δ^{-1} is the desired β. By transversality, we assume that γ is disjoint from the attaching circles of all the 1- and 2-handles of W_2 and then we push γ to the boundary M_2.

By turning handles upside down, we see that the inclusion induced map $\pi_1(M_2) \to \pi_1(W_2)$ is an isomorphism. Thus γ bounds an immersed disc in M_2, since it is null-homotopic in W_2. Since W_2 has dimension ≥ 5 we can assume that γ bounds an embedded disc in M_2. (This argument also works in ambient dimension four, see Exercise 2.5).

Thicken this disc to produce a cancelling 2-/3-handle pair. More precisely, insert a collar of $M_2 \times [0,1]$ into the handle decomposition and thicken by pushing the interior of the disc into this collar. The result is the addition of a single cancelling 2-/3-handle pair compatible with the old handle decomposition. By the choice of γ the 2-handle cancels the 1-handle h_1, leaving the 3-handle behind. Iterating this process allows us to trade all the 1-handles in W for 3-handles.

Step 2. *Use the Whitney trick to cancel all the other handles.*

This is the most important step in the argument. We will describe the Whitney trick in more detail in Section 3, with a focus on dimension four. In high dimensions, it was introduced by Whitney in [Whi44], where he used it to prove his embedding theorem, that every smooth, compact d-manifold embeds in $\mathbb{R}^d$.

Let M_2 denote the n-manifold obtained from M_0 after attaching all the 2-handles in W. Consider the chain complex $C_*(W, M_0; \mathbb{Z})$ given by the (latest) handle decomposition:

$$\longrightarrow C_4 \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_3} C_2 \longrightarrow 0$$

Since C_2 is free and $H_*(W, M_0; \mathbb{Z}) = 0$, the matrix for ∂_3 has the form $\partial_3 = \left[\frac{I_{p\times p}}{0_{p'\times p}}\right]$ for some p, p', where $I_{p\times p}$ is the $p \times p$ identity matrix, and $0_{p'\times p}$ is the $p' \times p$ matrix containing only zeros. On the other hand, basis changes can be effected by handle slides (corresponding to elementary row and column operations) and sign changes (corresponding to changing the orientation on individual handles). Therefore, we may assume that for each 2-handle h_2, there exists a unique 3-handle h_3 so that the belt sphere of h_2 and the attaching sphere of h_3, both contained in M_2, intersect algebraically once. If these intersected precisely once *geometrically*, we would be able to cancel the handles. The Whitney trick will tell us precisely why we may assume that these submanifolds do in fact intersect geometrically once, up to isotopy.

Let P^k and Q^{n-k} be transversely intersecting, smooth, compact, connected, oriented submanifolds of M_2^n, where M_2 is simply connected, oriented, and $n \geq 5$. Assume further that $\pi_1(M_2 \setminus (P \cup Q)) = 1$. We skip the proof of this final assumption for the moment, but rest assured this can be arranged in all the cases needed in the proof of Theorem 2.2. By our assumptions, we know that the intersections between P and Q are isolated double points, each equipped with a sign. Choose two intersection points of opposite sign. Choose arcs in P and Q joining the two double points. The union of these two arcs is called a *Whitney circle*. A disc bounded by a Whitney circle is called a *Whitney disc*. Since $\pi_1(M_2 \setminus (P \cup Q)) = 1$, there exists a Whitney disc D with interior in the complement of $P \cup Q$, which may be further assumed to be embedded since $n \geq 5$. Under a condition on the normal bundle of D in M_2 described in the next paragraph, we can push P along D and over, as indicated in Figure 1, to geometrically cancel the two algebraically cancelling intersection points. This process is called the *Whitney trick* or *Whitney move*.

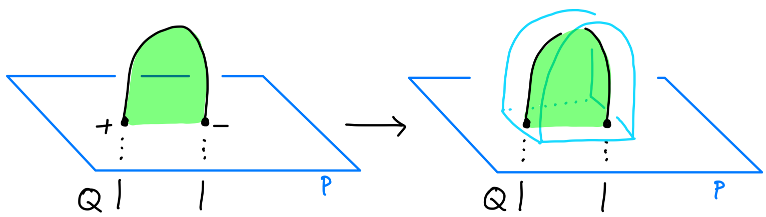


FIGURE 1. The Whitney move. Left: A Whitney disc D is shown in light green. Right: The Whitney move across D removes two intersection points.

We now describe the necessary condition on the normal bundle of D. Any embedded disc D with boundary a circle C pairing points in $P \pitchfork Q$ determines a $(k-1)$-dimensional sub-bundle of the normal bundle $\nu_{D \subseteq M_2}|_C$ of D restricted to C, by requiring that the sub-bundle be tangent to P

and normal to Q. In order to perform the Whitney trick we need this sub-bundle over the circle C to extend over the entire disc D. Standard bundle theory implies that the sub-bundle extends if and only if it determines the trivial element in $\pi_1(\mathrm{Gr}_{k-1}(\mathbb{R}^{n-2}))$, where the Grassmannian $\mathrm{Gr}_{k-1}(\mathbb{R}^{n-2})$ is the space of $(k-1)$-dimensional subspaces in $\mathbb{R}^{n-2}$. For $n-k \geq 3$, it is known that $\pi_1(\mathrm{Gr}_{k-1}(\mathbb{R}^{n-2})) \cong \mathbb{Z}/2$, and the nontrivial element corresponds to circles pairing intersection points with the same sign. In our current situation, we have $n \geq 5$ and $k \geq 2$, so at lease one of k or $k' = n-k$ will satisfy the codimension condition above. Since Whitney circles by definition pair intersection points of opposite sign, the sub-bundle in question extends, and we can perform the Whitney move.

To summarise, we previously knew that for each 2-handle h_2 in W, there exists a unique 3-handle h_3 so that the belt sphere of h_2 and the attaching sphere of h_3, both contained in M_2, intersect algebraically once. By the Whitney trick, we can assume, that the belt sphere of h_2 and the attaching sphere of h_3 intersect geometrically once, and therefore, all the 2-handles may be cancelled (using a subset of the 3-handles). But now the process can be iterated, by cancelling every k-handle using a subset of the $(k+1)$-handles. At the end of this process, there will be no remaining handles, showing that our original cobordism W is diffeomorphic to the product $M_0 \times [0,1]$, as desired. □

Remark 2.7. While we only applied handle trading to address 1-handles, the technique can also be used to tackle higher index handles. In [Wal71], Wall shows that by handle trading one can reduce the relative handle decomposition of an h-cobordism to consist of handles in precisely two consecutive indices. If we had used this method, we would have a particularly simple handle chain complex in Step 4 of the above proof.

The techniques in the above proof will be relevant to multiple proofs moving forward. To begin with, we now have most of the ingredients to prove Wall's half of the proof of Theorem 1.1, which we now state.

Theorem 2.8 (Wall [Wal64b, Theorem 2])**.** *Let M and N be smooth, closed, oriented, simply connected 4-manifolds with isomorphic intersection forms $Q_M \cong Q_N$. Then M and N are smoothly h-cobordant.*

Proof. Since $Q_M \cong Q_N$ we know that $\sigma(M \# -N) = 0$. Then by results of Rochlin and Thom [Roc52, Tho54] (see also [Mel84; Kir89, Chapter VIII]), there exists a smooth, compact, oriented W^5, with $\partial W = -N \sqcup M$. More specifically, this uses that the oriented bordism group $\Omega_4^{SO} \cong \mathbb{Z}$. We will gradually modify W until it becomes an h-cobordism. However, we warn the reader that, unlike the previous proof, the modifications will often change the diffeomorphism type of W.

Step 1. *Perform surgery on circles to modify W to be simply connected.*

Choose smooth, embedded loops $\alpha_1, \ldots, \alpha_k$ in the interior of W which normally generate $\pi_1(W)$. Such a finite list of homotopy classes exists since W is compact, and they can be represented by embedded loops because of transversality. Perform surgery on each α_i, i.e. remove a neighbourhood $\alpha_i \times D^4$ and glue in $D^2 \times S^3$ for each i, using that $\partial(\alpha_i \times D^4) \cong S^1 \times D^4 = \partial(D^2 \times S^3)$. The framing does not matter – all we need is that the copies of $D^2 \times \{*\}$ glued in provide null homotopies of the curves $\{\alpha_i\}$. By construction, the result, denoted V, is simply connected.

Step 2. *Arrange that V is built by attaching only* 2- *and* 3-*handles along $N \times [0,1]$.*

We use the techniques from the proof of Theorem 2.2: Begin with a handle decomposition for V relative to N. As before, cancel the 0-handles, and then perform handle trading to trade 1-handles for 3-handles. The latter step uses that $\pi_1(V)$ is trivial. Now turn the handle decomposition upside down, and repeat the two previous steps, then turn right side up again. The result is a handle decomposition of V consisting only of 2- and 3-handles.

Step 3. *Observe that the middle level $M_{1/2} \cong N\#m(S^2 \times S^2) \cong M\#m(S^2 \times S^2)$ for some m.*

Let $M_{1/2}$ denote the 4-manifold obtained by attaching the 2-handles of V to $N \times \{1\} \subseteq N \times [0,1] \subseteq V$. Observe that attaching a 2-handle changes the 4-manifold by surgery on an embedded circle; in other words, for a single 2-handle attachment, we would have $M_{1/2} = N \setminus (\alpha \times D^3) \cup (D^2 \times S^2)$, where $\alpha \times D^3$ is the attaching region of the 2-handle, and $D^2 \times S^2$ is the belt region. Since N is simply connected, the attaching circles for the 2-handles are null-homotopic in N, and bound embedded discs $\{\Delta_i\}$ (see Exercise 2.5). Summarising, $M_{1/2}$ is produced from N by surgery on trivial unknotted circles. This produces either $S^2 \times S^2$ or $S^2 \widetilde{\times} S^2$ summands [Wal99, Lemma 5.5]. In each $S^2 \times S^2$ summand, the factor $\{*\} \times S^2$ corresponds to the belt sphere $\{*\} \times S^2 \subseteq D^2 \times D^3$ for the corresponding 2-handle. The factor $S^2 \times \{*\}$ is the union of the disc Δ_i with a pushoff of the core $D^2 \times \{*\}$ of the 2-handle. There are two possible framings for the 2-handle attachment, corresponding precisely to the choice of $S^2 \times S^2$ or $S^2 \widetilde{\times} S^2$.

We now argue that it is possible to only have $S^2 \times S^2$ summands. In case that N is non-spin, we use Exercise 2.10 [Wal64a, Corollary 1] (see also [GS99, Proposition 5.2.4]). Note also that a simply connected 4-manifold is spin if and only if the intersection form is even [GS99, Remark 1.4.27(c)], so N is non-spin if and only if M is non-spin, since $Q_M \cong Q_N$ and both are simply connected. In the case that N (and equivalently M) is spin, we should have started the proof assuming W is spin, using the fact that $\Omega_4^{\text{Spin}} \cong \mathbb{Z}$ (see [Wal64b, Lemma 1; Kir89, Chapter VIII]), in which case it is not hard to see that the 2-handles must be attached using the framing which produces $S^2 \times S^2$ summands.

The same argument applied to the upside down handle decomposition shows that $M_{1/2} \cong M\#m'(S^2 \times S^2)$ where m' is the number of 3-handles. Then we know that $N\#m(S^2 \times S^2) \cong M_{1/2} \cong M\#m'(S^2 \times S^2)$ and $Q_M \cong Q_N$, so $m = m'$.

Step 4. *Cut up and reglue with a twist.*

In this final step, we will need the following result of Wall.

Theorem 2.9 (Wall [Wal64a, Theorem 2])**.** *Let P be a smooth, closed, simply connected, oriented 4-manifold with Q_P indefinite. Then every automorphism of $Q_{P\#S^2\times S^2}$ can be realised by a self-diffeomorphism of $P\#S^2 \times S^2$.*

We want to cut V along the middle level $M_{1/2}$ and then reglue using an appropriate self-diffeomorphism, so that the result is an h-cobordism. Since V, M, and N are simply connected (and the result of regluing will also be simply connected), it suffices to control the relative homology of the result (see Exercise 2.1). Observe that $Q_{M_{1/2}}$ is indefinite as long as $m \geq 1$ (if $m = 0$, there are no handles in V, and therefore, V is a (trivial) h-cobordism). Theorem 2.9 will apply to $M_{1/2}$ as long as $m \geq 2$, or Q_N is indefinite and $m \geq 1$. But we can arrange for $m \geq 2$ by adding a cancelling 2-/3-handle pair to the handle decomposition of V. It remains only to choose a suitable automorphism of $Q_{M_{1/2}}$.

Note that $H_2(M_{1/2};\mathbb{Z}) \cong H_2(N;\mathbb{Z}) \oplus \mathbb{Z}\langle \alpha_1, \overline{\alpha}_1, \dots, \alpha_m, \overline{\alpha}_m\rangle$, where each α_i corresponds to the core of a 2-handle and each $\overline{\alpha}_i$ to a belt sphere. Similarly, looking at the upside down handle decomposition, we see that $H_2(M_{1/2};\mathbb{Z}) \cong H_2(M;\mathbb{Z}) \oplus \mathbb{Z}\langle \beta_1, \overline{\beta}_1, \dots, \beta_m, \overline{\beta}_m\rangle$, where each β_i corresponds to the belt sphere of an upside down 3-handle (i.e. the attaching sphere of the

3-handle) and each $\overline{\beta}_i$ to the core of an upside down 3-handle. By hypothesis, there exists an isomorphism $\varphi\colon H_2(M;\mathbb{Z}) \to H_2(N;\mathbb{Z})$ inducing the isomorphism $Q_M \cong Q_N$. Extend φ to $\overline{\varphi}\colon H_2(M_{1/2};\mathbb{Z}) \to H_2(M_{1/2};\mathbb{Z})$ by setting $\beta_i \mapsto \alpha_i$ for each i. Then $\overline{\varphi}(\overline{\beta}_i) = \overline{\alpha}_i$ by unimodularity of the intersection form, since, for example, $1 = Q(\beta_i, \overline{\beta}_i) = Q(\overline{\varphi}(\beta_i), \overline{\varphi}(\overline{\beta}_i))) = Q(\alpha_i, \overline{\varphi}(\overline{\beta}_i)) = Q(\alpha_i, \overline{\alpha}_i)$.

Now consider the diffeomorphism $\widetilde{\varphi}\colon M_{1/2} \to M_{1/2}$ corresponding to $\overline{\varphi}$ provided by Theorem 2.9. Specifically, let V_2 denote the union of $N \times [0,1]$ with the 2-handles of V, and let V_3 denote the union of the 3-handles of V. Let V' denote the result of gluing V_2 and V_3 along their common boundary $M_{1/2}$ via the diffeomorphism $\widetilde{\varphi}$. By construction, for each 2-handle h_2, there exists a unique 3-handle h_3 such that the belt sphere of h_2 intersects the attaching sphere of h_3 algebraically once; and all other attaching spheres of 3-handles intersect the belt sphere of h_2 algebraically zero times. As a result, V' is an h-cobordism. □

The above proof also applies to the following theorem.

Theorem 2.10 (Wall [Wal64b, Theorem 3])**.** *Let M and N be smooth, closed, simply connected, oriented, h-cobordant 4-manifolds, then there exists m so that $M\#m(S^2 \times S^2) \cong N\#m(S^2 \times S^2)$.*

Proof. An h-cobordism between spin 4-manifolds is in particular a spin cobordism. Apply Steps 2 and 3 of the previous proof. □

As a side note, we now have many of the tools necessary to prove the following theorem.

Theorem 2.11 (Curtis-Hsiang-Freedman-Stong [CFHS96], Matveyev [Mat96], Bižaca (unpublished), see also Kirby [Kir96])**.** *Let M_0 and M_1 denote smooth, closed, simply connected, oriented 4-manifolds and W a smooth, compact, oriented h-cobordism from M_0 to M_1. Then there exists a compact sub-h-cobordism $A \subseteq W$ between $A_i \subseteq M_i$ such that*

(i) *$W \setminus \operatorname{Int} A$ is a product h-cobordism, i.e. there is a diffeomorphism*
$$W \setminus \operatorname{Int} A \cong (M_0 \setminus \operatorname{Int} A_0) \times [0,1],$$
restricting to the identity on $M_0 \setminus \operatorname{Int} A_0$; and

(ii) *$A_0 \cong A_1$ is contractible.*

Proof sketch. As before, we manipulate a handle decomposition of W until it consists only of 2- and 3-handles. In this case we know that the belt spheres of the 2-handles and the attaching spheres of the 3-spheres intersect so that $H_*(W, M_0;\mathbb{Z}) = 0$. The sub-$h$-cobordism A will contain all the handles of W, so that the complement will necessarily be a product. However, A will contain some more material, in particular enough material to make it contractible. □

As a corollary of the above theorem, we have the following cork theorem. See the Piccirillo mini-course for more on corks.

Theorem 2.12 (Cork theorem)**.** *Any two smooth, closed, simply connected, homeomorphic 4-manifolds M_0 and M_1 are related by a* cork twist*, i.e. M_1 is obtained from M_0 by removing some compact, contractible submanifold $C \subseteq M_0$ and regluing via a diffeomorphism.*

Proof. By Theorem 2.8, there is a smooth h-cobordism W from M_0 to M_1. Then by Theorem 2.11, there is the required decomposition, where we use $C = A_0$. □

Exercises for Lecture 1

Introductory problems.

Exercise 2.1. Prove that a smooth, compact, oriented $(n+1)$-manifold with $\partial W = -M_0^n \sqcup M_1^n$, with W, M_0, and M_1 simply connected is an h-cobordism if and only if $H_*(W, M_0; \mathbb{Z}) = 0$.

Exercise 2.2. Prove Corollary 2.3: Let $n \geq 6$. Every smooth homotopy n-sphere is homeomorphic to S^n. Here a *homotopy n-sphere* is a manifold which is homotopy equivalent to S^n. Does your proof give a diffeomorphism in the output? Look up in which dimensions the smooth Poincaré conjecture is still open.

Exercise 2.3. Prove that modifying the handle attaching maps in a smooth handle decomposition by smooth isotopy preserves the diffeomorphism type of the resulting manifold.

Exercise 2.4. Compute the fundamental group and homology of a smooth manifold in terms of its handle decomposition.

Exercise 2.5. Let γ be an embedded circle in the interior of a smooth manifold W^m, with $m \geq 4$ and $\pi_1(W) = 1$. Prove that γ bounds an embedded disc in the interior of W.

Moderate problems.

Exercise 2.6. Formulate a definition of a relative h-cobordism between smooth n-manifolds with diffeomorphic boundary.

Exercise 2.7. Prove that a topological 4-manifold has a topological handle decomposition if and only if it is smoothable.

Exercise 2.8. Prove that relative handle decompositions exist.

Exercise 2.9. Can we use the Whitney trick to cancel 1-handles in Smale's proof of Theorem 2.2?

Exercise 2.10. Let N^4 be closed, smooth, simply connected, and non-spin. Then $N\#(S^2 \times S^2) \cong N\#(S^2 \widetilde{\times} S^2)$.

Challenge problems.

Exercise 2.11. Prove that (locally finite, smooth) handle decompositions exist for noncompact, smooth manifolds.

3. The Whitney trick in dimension four and the disc embedding theorem

3.1. **Visualising surfaces in dimension four.** Recall that by transversality, surfaces in a 4-manifold may be assumed, up to isotopy, to intersect in isolated double points. In this section, we briefly describe how to visualise such intersections.

The model for intersections of surfaces in a 4-manifold is the intersection between the xy- and zt-planes in $\mathbb{R}^4$, considered as the product of $\mathbb{R}^3$ with a copy of $\mathbb{R}$ given by a time coordinate t. In Figure 2, we see $\mathbb{R}^4$ depicted as a sequence of $\mathbb{R}^3$ slices, each corresponding to a different value of t. The xy-plane appears in the $t = 0$ slice. The zt-plane appears in each slice as a (vertical) line. As we go backwards and forwards in time, these lines trace out the zt-plane. As expected, the two planes intersect at precisely one point, namely the origin.

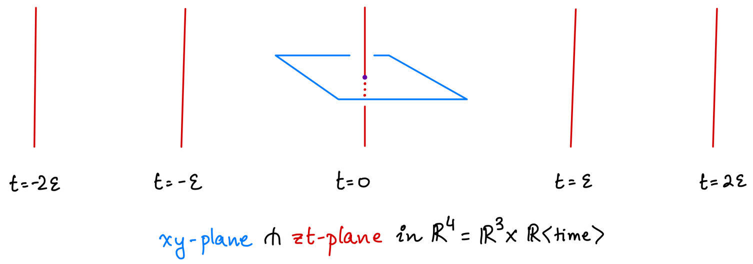


Figure 2. $\mathbb{R}^4$ is depicted as a sequence of $\mathbb{R}^3$ slices, each corresponding to a different value of t. The blue plane in the central $t = 0$ slice depicts the xy-plane. The red vertical lines trace out the zt-plane. The unique point of intersection of the two planes is at the origin $0 \in \mathbb{R}^4$, as expected. This picture provides a local model for transverse intersections between surfaces in an arbitrary 4-manifold.

Transverse intersections between surfaces in an arbitrary 4-manifold are locally modelled by Figure 2, that is, given surfaces P and Q in a 4-manifold M, and a point $x \in P \pitchfork Q$, there is a neighbourhood of x in M which is homeomorphic to $\mathbb{R}^4$, in which the planes P and Q appear as in Figure 2.

Given surfaces P and Q intersecting transversely at a point $x \in M$, let B denote a small ball around x, small enough so that Figure 2 models P and Q within B. The boundary ∂B is a copy of S^3. By transversality, we may assume that the intersection $\partial B \cap (P \cup Q)$ is a 1-manifold. Indeed, this 1-manifold is a Hopf link in $\partial B \cong S^3$ (see Exercise 5.1 and Figure 3). Each component of this link is a meridian of either P or Q. By definition, this means that each bounds a disc that intersects precisely one of P or Q at a single point.

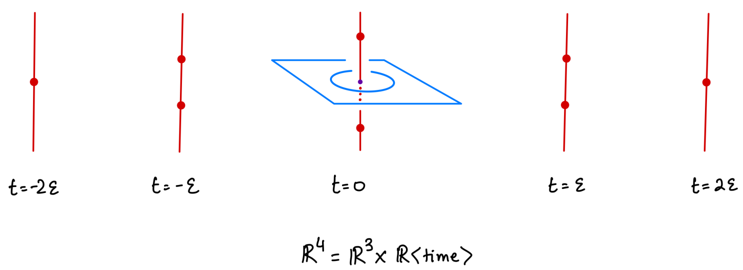


Figure 3. The blue circle and the red dots show the intersection between $P \cup Q$ (shown in blue and red respectively) and the boundary of a small ball centred at the origin of the image. These circles form a Hopf link according to Exercise 5.1. The blue circle is a meridian of Q and the red dots form a meridian of P.

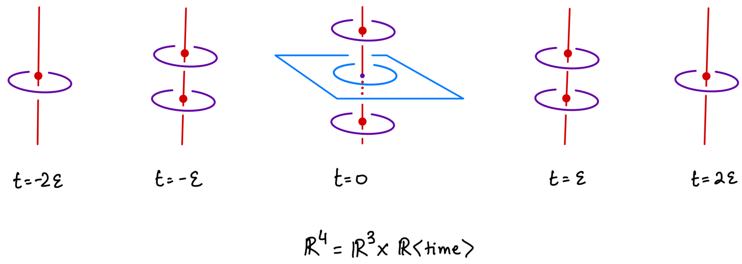


FIGURE 4. The Clifford torus (purple) around a point $x \in P \pitchfork Q$, for surfaces P (blue) and Q (red) in a 4-manifold M.

Given surfaces P and Q intersecting transversely at a point $x \in M$, the Clifford torus at x is the product of the two meridians of P and Q mentioned above (see Figure 4).

3.2. **The Whitney trick in dimension four.** Recall that the *Whitney trick* was a key ingredient of the proof of Theorem 2.2. We want to understand to what extent the Whitney trick is available in ambient dimension four. Consider the situation in Figure 5. Blue and red depict oriented surfaces P and Q respectively, in some ambient, smooth, oriented 4-manifold M, intersecting in two points p and q with opposite sign. Choose an embedded arc γ in P from p to q and an embedded arc δ in Q from p to q where the union $C := \gamma\delta^{-1}$ bounds an embedded disc D whose interior lies in the complement of f and g. In the ideal situation of Figure 5, these three items are visible in the central $t = 0$ time slice.

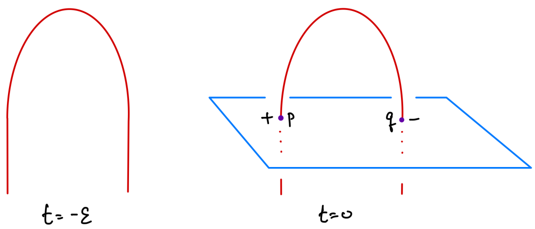


FIGURE 5

Recall that P, Q, and M are all oriented. The orientations of P and Q determine an orientation of T_pM and T_qM. Comparing with the given orientation on T_pM and T_qM (coming from the orientation on M), we get a function $\operatorname{sgn}\colon \{p, q\} \to \{+, -\}$. The normal bundle of D in M is a trivial 2-plane bundle. Fix an orientation on the fibres. Consider the following 1-plane sub-bundle V of the normal bundle of D restricted to $C = \partial D = \gamma\delta^{-1}$. The sub-bundle along γ is given by the tangent bundle to P. This can be extended to a choice of sub-bundle along δ that is normal to P and agrees with TP at p and q, since the intersections are transverse. This is a trivial 1-plane bundle if and only if the function $\operatorname{sgn}\colon \{p, q\} \to \{+, -\}$ is surjective, that is, if and only if the signs of p and q are opposite.

Assuming that this is the case, choose a section s of the sub-bundle V. Since V is 1-dimensional, the section s is determined up to multiplication by a continuous function $S^1 \to \mathbb{R} \setminus \{0\}$. We say that the Whitney disc D is *framed* if the section s extends to a nonvanishing section on the normal bundle of all of D in M. The framing of the normal bundle of D restricted to ∂D, induced by s and the chosen orientation on the fibres of the normal bundle, is called the *Whitney framing*.

In general, the 'tangent to one sheet, normal to the other' principle gives the desired framing on C. The framing induced by a candidate Whitney disc may differ from this; since $\pi_1(\mathrm{Gr}_1(\mathbb{R}^2)) \cong \mathbb{Z}$, we have a *twisting number* or *relative Euler number* in $\mathbb{Z}$.

Extend the Whitney disc D very slightly beyond its borders; more precisely, extend γ slightly beyond p and q in P and push δ out along the radial direction of $TD|_\delta$ i.e. the direction orthogonal to $T\delta$. Now consider the disc bundle $DE \cong D^2 \times D^1$, which is the sub-bundle of the normal bundle of (the extended version of) D determined by the section s, where D coincides with the zero section. The boundary of DE is a 2-sphere, with $\partial(DE) \cap P$ a neighbourhood of γ, that we denote by S. The Whitney move pushes the strip S across DE. The outcome has S replaced by two parallel copies of the Whitney disc D together with a strip whose core is parallel to δ. This is an isotopy of the surface P (if $P \neq Q$), and a regular homotopy of $P \cup Q$. The latter fact holds since we have described a homotopy through local embeddings. Note that we used a framed and embedded Whitney disc with interior in the complement of $P \cup Q$, and the two intersection points p and q were removed, as desired.

In the case that D is framed, but not embedded, or the interior intersects $P \cup Q$, the Whitney move, now called a (*framed*) *immersed Whitney move*, still uses the same strip S in a neighbourhood of δ, and two copies of D obtained using s and $-s$, where s is a section of the normal bundle. The resulting move is a regular homotopy of P, and not an isotopy, even if $P \neq Q$. In particular, the intersection points p and q are removed by an immersed Whitney move, but four new self-intersection points of P are created for each self-intersection point of D, and two new intersections of $P \cup Q$ are created for each intersection of the interior of D with $P \cup Q$.

In more generality, if D intersects a surface Σ, where Σ may equal P or Q, but need not, then two intersection points of P with Σ are created for each intersection point of D with Σ.

3.3. **When can we perform the Whitney trick?** Let P and Q be oriented surfaces intersecting transversely at points p and q with opposite sign, within a smooth, oriented, simply connected ambient 4-manifold M. Choose arcs γ and δ as above, and let C denote the loop $\gamma\delta^{-1}$. Since M is smooth and simply connected, we can find an immersion $\Delta \colon D^2 \looparrowright M$ with boundary C. In general, we will wish to perform an isotopy of P with the result of removing the two (algebraically cancelling) intersection points p and q. To do so, we need to find an embedding $\overline{\Delta} \colon D^2 \to M$ such that the following three conditions are satisfied.

(1) $\overline{\Delta}$ is an embedding.
(2) $\overline{\Delta}(\operatorname{Int} D^2) \cap (P \cup Q) = \emptyset$.
(3) $\overline{\Delta}(D^2)$ is framed, meaning that the relative Euler number is trivial.

We next consider when the above conditions are satisfied.

Remark 3.1. For the purposes of these lectures, we will restrict ourselves to simply connected ambient 4-manifolds as much as possible. However, the theory does apply to 4-manifolds with more general fundamental groups, where it may not be immediate that Whitney circles are null-homotopic. In these settings one needs a more precise algebraic count of intersections, called the *equivariant* intersection and self-intersection numbers. See for example Exercise 5.3. The theory also applies to non-orientable 4-manifolds, in which case we must begin with a specified orientation at a given basepoint and transport these along specified paths to determine the signs of intersection points of surfaces. We will avoid talking about this more general setting as much as possible; the interested reader is directed to [PRT20, KPRT21].

3.3.1. *Boundary twisting.* Suppose we have two immersed surfaces A and B in a 4-manifold M such that part of the boundary of B is embedded in A, as shown in Figure 6, and this part of ∂B lies in the interior of M, for example, when B is a Whitney disc pairing intersection points of A with itself or some other surface. The operation of *boundary twisting* B about A consists of changing a collar of B near a point in its boundary on A, as depicted in Figure 6. Note that this creates a new point of intersection between A and B and changes the framing of B by a full twist. The upshot of this paragraph is that given an immersed disc with boundary C, we may modify it using boundary twists so that the result is framed.

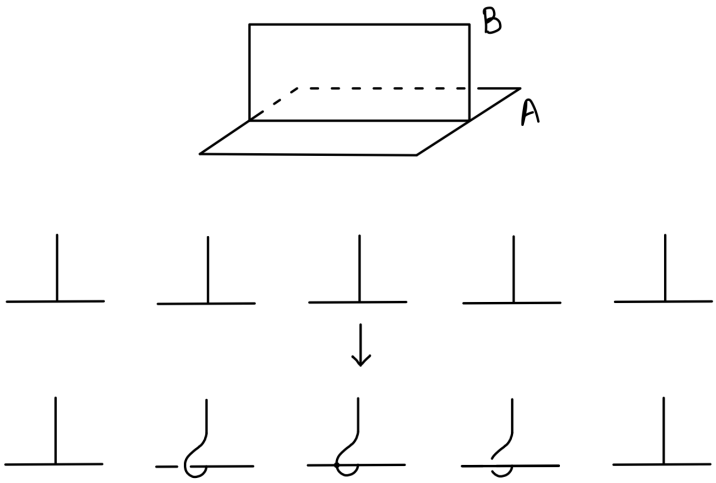


FIGURE 6. Top: Surfaces A and B in a 4-manifold M such that part of the boundary of B is embedded in A. Middle: Cross sections of the picture on top, before boundary twisting. Bottom: Cross sections of A and B, after boundary twisting. Note the new point of intersection between A and B in the middle cross section.

3.3.2. *π_1-negligibility.* In general, if $\pi_1(M \setminus (P \cup Q)) = 1$ we may assume that the image of $\mathrm{Int}(D^2)$ under Δ lies in the complement of $P \cup Q$. This occurs precisely when $P \cup Q$ has *geometrically dual spheres*, i.e. immersed spheres $P^\perp, Q^\perp \subseteq M$ so that $P^\perp \pitchfork P$ a point and $P^\perp \pitchfork Q = \emptyset$, as well as $Q^\perp \pitchfork Q$ a point and $Q^\perp \pitchfork P = \emptyset$. However, there is a problem here: namely, this condition does not guarantee that Δ is framed. We can use boundary twisting to correct the framing, but this will introduce new intersections between the interior of Δ and P or Q.

The best compromise in this situation is to ask for *framed* geometrically dual spheres, i.e. we assume that $P^\perp$ and $Q^\perp$ above have trivial normal bundle. In this case, we can start with an arbitrary $\Delta\colon D^2 \looparrowright M$, boundary twist to correct the framing, and then tube into $P^\perp$ and $Q^\perp$ as appropriate (see Figure 7) to ensure that the interior of Δ lies in the complement of $P \cup Q$. See Exercise 5.2.

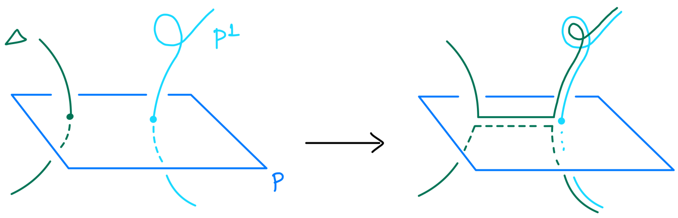


FIGURE 7. Suppose that $P^\perp$ intersects P precisely once. Then given any intersection between Δ and P, we may *tube* it into $P^\perp$. More precisely, we perform an ambient connected sum of Δ and a pushoff of $P^\perp$ using the meridional annulus of an embedded arc on P.

3.3.3. *Embedding discs.* The discussion so far shows that we can find a framed, immersed Whitney disc Δ for C with interior in the complement of $P \cup Q$. This leaves us with the question: when can we improve an properly immersed disc in a 4-manifold to an embedded disc, relative to the boundary?

Unfortunately, not every proper, immersed disc in a 4-manifold can be replaced by an embedded disc, relative to the boundary. To see this, note that every knot $K \subseteq S^3$ is the boundary of a proper, immersed disc – this follows directly from the fact that $\pi_1(B^4) = 1$, or alternatively, we see a concrete proper, immersed disc given by the cone on the knot. A knot which bounds a smooth, proper embedded disc in B^4 is said to be *smoothly slice* [FM66]. Not all knots are smoothly slice, for example, the trefoil. See the Piccirillo mini-course for more on slice knots.

However, in certain cases immersed discs may be promoted to embedded discs. This is precisely the content of the celebrated disc embedding theorem, which we now state in a preliminary form. We will discuss a far more general form later in the lectures.

Theorem 3.2 (Disc embedding theorem, preliminary version [Cas86, Fre82, FQ90])**.** *Let M be a smooth, oriented, simply connected 4-manifold. Suppose we have a map*

$$\begin{array}{ccc} D^2 & \xrightarrow{f} & M \\ \uparrow & & \uparrow \\ \partial D^2 & \hookrightarrow & \partial M, \end{array}$$

where $f|_{\partial D^2}$ is a flat embedding. Suppose further that there is an immersion $g\colon S^2 \looparrowright M$ such that the algebraic intersection number $\lambda(f,g) = 1$, the algebraic self-intersection number $\mu(g) = 0$, and $g(S^2)$ has trivial normal bundle.

Then there exists a flat embedding $\overline{f}\colon D^2 \hookrightarrow M$ with $\overline{f}|_{\partial D^2} = f|_{\partial D^2}$, and inducing the same framing on the boundary.

In the statement above, the intersection number $\lambda(f,g)$ is the signed count of intersections between $f(D^2)$ and $g(S^2)$. Similarly the self-intersection number $\mu(g)$ is the signed count of self-intersections of $g(S^2)$. Both of these quantities have analogues for non-simply connected M; see for example Exercise 5.3.

3.4. **Contraction and push off.** Let Σ be a closed surface in a 4-manifold M. A *cap* for Σ is a (potentially immersed) disc in M with boundary a homologically essential simple closed curve on Σ. We insist that the surface induced framing matches the framing on the boundary of the cap induced by its normal bundle. A *capped surface* is an embedded surface Σ in M, along with its normal bundle, and a collection of caps attached to a symplectic basis for curves for the first homology of Σ.

The (symmetric) contraction of a capped surface, depicted in Figure 8, converts a capped surface into an immersed sphere. As shown in the figure, we surger the surface using two copies of each cap, joined by a square at the points of intersection of the boundaries. One could alternatively contract a capped surface by only surgering along one disc per dual pair (this would then be called *asymmetric* contraction, see Figure 9), but this would not enable the pushing off procedure that we are about to describe in the next paragraph. Henceforth, whenever we talk about contraction, by default we will mean the symmetric contraction. Observe that the result of contracting a capped surface has algebraically cancelling self-intersections, in particular since we insisted on the framings matching up correctly.

After contracting a capped surface Σ^c with body Σ, any other surface $A \subseteq M$ that intersected the caps of Σ^c can be pushed off the contracted surface, as we describe in Figure 10. The fact that we can perform the pushing off procedure, which is a regular homotopy, shows that the intersection number of the contracted surface with A agrees with the intersection number

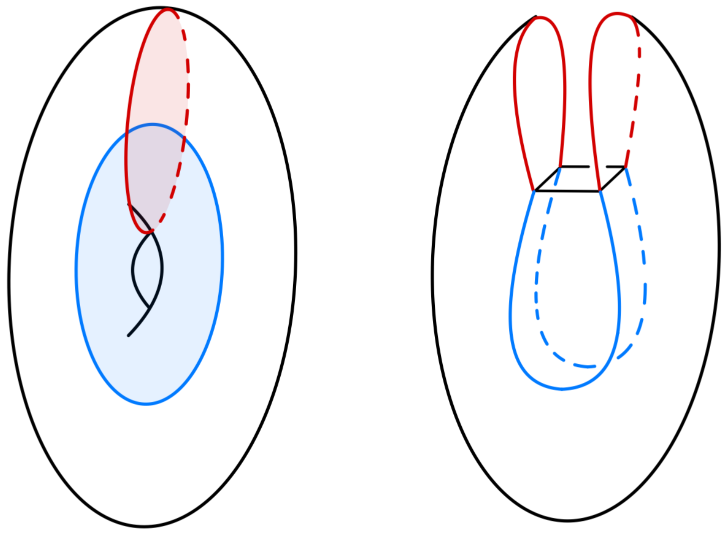


FIGURE 8. Left: A capped surface with embedded caps. Only the 2-skeleton is shown. Right: The result of (symmetric) contraction.

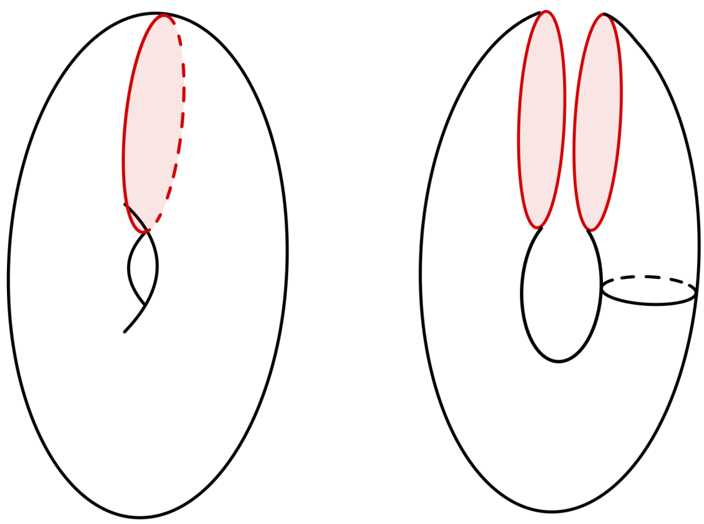
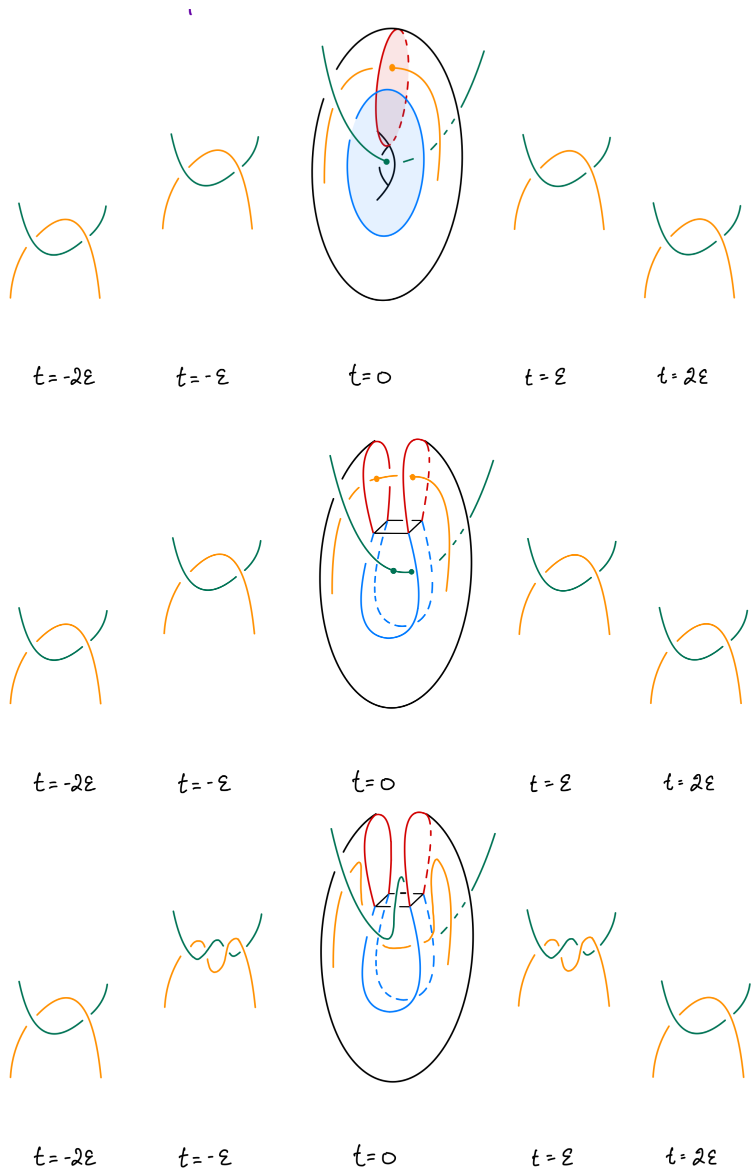


FIGURE 9. Left: A surface with an embedded cap. Only the 2-skeleton is shown. Right: The result of (asymmetric) contraction. Warning: this is *not* what we will mean by a contraction in the sequel.

of A with the (uncapped) surface Σ, i.e. there is no new contribution from the caps. (A key observation is that this latter fact holds regardless of the ambient 4-manifold or its fundamental group.) The push off procedure reduces the number of intersection points between the contracted surface (an immersed sphere) and the pushed off surfaces, so we gain some disjointness at the expense of converting a capped surface into an immersed sphere. An additional cost is as follows. Suppose that a surface A intersects a cap of the capped surface, and a surface B intersects a dual cap. Then after pushing both A and B off the contraction, we obtain two new (algebraically cancelling) intersection points between A and B.


$t = -2\varepsilon$
$t = -\varepsilon$
$t = 0$
$t = \varepsilon$
$t = 2\varepsilon$
$t = -2\varepsilon$
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$t = \varepsilon$
$t = 2\varepsilon$
$t = -2\varepsilon$
$t = -\varepsilon$
$t = 0$
$t = \varepsilon$
$t = 2\varepsilon$

FIGURE 10

4. Freedman's h-cobordism theorem

We will outline the proof of the following theorem.

Theorem 4.1 (Category losing h-cobordism theorem [Fre82])**.** *Let W^5 be a smooth, compact, oriented, simply connected h-cobordism from M_0^4 to M_1^4. Then $W \approx M_0 \times [0,1]$.*

More specifically, there exists a homeomorphism $\varphi\colon W \to M_0 \times [0,1]$, where the restriction $\varphi|_{M_0}\colon M_0 \to M_0$ is the identity map. Note that the restriction $\varphi|_{M_1}$ is a homeomorphism from M_1 to M_0.

Remark 4.2. Unlike in Theorem 2.2, the above theorem only asserts the existence of a homeomorphism rather than a diffeomorphism. Indeeed, the work of Donaldson shows that there is no analogous category preserving statement, i.e. there exist smooth, compact, oriented, simply connected h-cobordisms that are not smoothly trivial. This was first shown by Donaldson in [Don87], specifically that $\mathbb{CP}^2 \# 9 \overline{\mathbb{CP}2}$ and its $(2,3)$ logarithmic transform, called the *Dolgachev surface*, are not diffeomorphic. As closed, smooth, simply connected, oriented 4-manifolds, they are smoothly h-cobordant by Wall's theorem (Theorem 2.8). By Freedman's theorem (Theorem 4.1), they are also homeomorphic. In particular, this implies the existence of a smooth, compact, simply connected, oriented h-cobordism between closed 4-manifolds which is homeomorphic, but not diffeomorphic, to a product. Particularly nice examples of such h-cobordisms have also been found, consisting of a single 2-handle and 3-handle each. The first example was found by Akbulut in [Akb91], between blowups of the $K3$ surface and its logarithmic 0-transform, and other examples of such h-cobordisms between blowups of elliptic surfaces and their logarithmic 0-transforms were constructed by Bižaca and Gompf in [BG96].

As previously mentioned, the combination of Theorem 2.8 and the above theorem proves Theorem 1.1. As a special case, we have the following category losing case of the Poincaré conjecture.

Corollary 4.3. *Every smooth homotopy 4-sphere is homeomorphic to S^4.*

In a subsequent lecture, we will consider the corresponding fully topological Poincaré conjecture in dimension four, i.e. that topological homotopy 4-spheres are homeomorphic to S^4 as well.

Outline of the proof of Theorem 4.1. As in the proof of Theorem 2.8, we can assume there is a handle decomposition of W relative to M_0 with only 2- and 3-handles. For simplicity, primarily of notation, we assume henceforth that there is a single 2-handle and a single 3-handle (and no other handles) in the decomposition. Let $M_{1/2}$ denote the middle-level of W, i.e. the 4-manifold produced after adding the 2-handle to $M_0 \times \{1\}$, or equivalently, by adding the upside down 3-handle (i.e. a 2-handle) to M_1. As before, we know that $M_{1/2} \cong M_0 \#(S^2 \times S^2) \cong M_1 \#(S^2 \times S^2)$. Let P denote the belt sphere of the 2-handle and Q denote the attaching sphere of the 3-handle. Note that $P, Q \subseteq M_{1/2}$, where $M_{1/2}$ is smooth, closed, simply connected, and oriented. Since W is an h-cobordism, we know that $\lambda(P,Q) = 1$. This means that P and Q intersect algebraically once (counted up to sign), but may intersect geometrically many more times. We now make another simplifying assumption: assume that P and Q intersect geometrically three times, i.e. there is one extraneous pair of intersections. Our goal is to isotope Q so that $P \pitchfork Q$ is (geometrically) a single point. If we achieve this, we will be able to cancel the 2-handle and the 3-handle in W, indicating that W is a trivial cobordism, that is, a product. Spoiler alert: we will be able to do this via a merely topological isotopy, indicating that W is *homeomorphic* (but not necessarily diffeomorphic) to the product $M_0 \times [0,1]$.

As a preliminary step, we prove the following lemma.

Lemma 4.4. *There exist framed, immersed spheres $P^\perp$, $Q^\perp \subseteq M_{1/2}$ so that $P^\perp \pitchfork P$ a point and $P^\perp \pitchfork Q = \emptyset$, as well as $Q^\perp \pitchfork Q$ a point and $Q^\perp \pitchfork P = \emptyset$.*

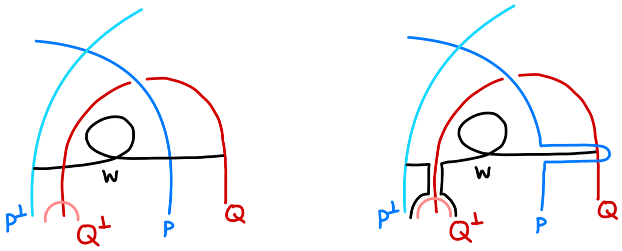


FIGURE 11

Proof. From our previous discussion, we know that there exist framed, embedded spheres $P^\perp$ and $Q^\perp$ in $M_{1/2}$ so that $P \pitchfork P^\perp$ is a single point and $Q \pitchfork Q^\perp$ is a single point (these are respectively the dual spheres $S^2 \times \{\text{pt}\}$ in the decomposition $M_{1/2} \cong M_0 \# S^2 \times S^2 \cong M_1 \# S^2 \times S^2$). The problem is that we have no control over how $P^\perp$ interacts with Q, and how $Q^\perp$ interacts with P. We will now ensure they interact the way we wish, at the expense of changing them from being embeddings to immersions.

First we work with $P^\perp$. Tube every point of intersection between $P^\perp$ and Q into parallel copies of P. That is, we repeatedly perform an ambient connected sum of $P^\perp$ and an appropriately oriented copy of P inside $M_{1/2}$ along a suitable arc. This might increase the number of intersections between $P^\perp$ and $Q^\perp$, or make $P^\perp$ immersed, but we do not mind. Now all the intersections between $P^\perp$ and Q can be paired by Whitney discs in $M_{1/2}$. Consider some such framed, immersed Whitney disc W. If we perform the Whitney move on $P^\perp$ along W right now we would be in danger of creating new intersections of $P^\perp$ with whatever W intersects, which *a priori* might be any of P, Q, $P^\perp$, or $Q^\perp$. However, we do not mind intersections between $P^\perp$ and $Q^\perp$ nor self-intersections of $P^\perp$. So the only problems are caused by intersections of W with P or Q.

We can remedy the Q intersections by tubing W along Q into push-offs of $Q^\perp$, where the push-offs use sections of the normal bundle transverse to the 0-section. This may lead to new intersections of W with P. Consequently, the new W only has problematic intersections with P which, in turn, can be removed by isotoping P off W by finger moves in the direction of Q, as shown in Figure 11. (By switching perspective, we think of these as isotopies of Q rather than of P.) At this point, we have possibly made the new Whitney disc more singular (if $Q^\perp$ meets W then tubing W into $Q^\perp$ creates new self-intersections of W) and created new (algebraically cancelling) intersections between P and Q, but this does not worry us for now. A Whitney move on $P^\perp$ along the new (framed) W produces a (probably immersed) geometric dual for P away from Q, as needed. We still call the result $P^\perp$.

By applying a similar process, we can upgrade $Q^\perp$ to a framed, immersed geometrically dual sphere for Q which does not intersect P, as claimed. □

By the discussion in Section 3.3, we see that there is a framed, immersed Whitney disc Δ for the extraneous pair of intersections between P and Q, with interior lying in $M_{1/2} \setminus (P \cup Q)$. We wish to apply the disc embedding Theorem 3.2 to Δ, so we check all the hypotheses. By Exercise 5.2, we know that $\pi_1(M_{1/2} \setminus (P \cup Q))$ is trivial, due to the existence of $P^\perp$ and $Q^\perp$ from Lemma 4.4. Since P and Q are smooth, $M_{1/2} \setminus (P \cup Q)$ is a smooth 4-manifold. The only missing ingredient is the algebraically dual sphere. This will come from the Clifford torus Σ at either of the two double points paired by Δ, see Figure 12.

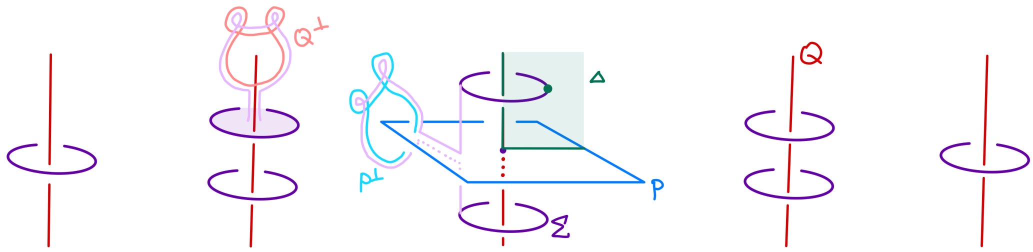


FIGURE 12

Caps for Σ are provided by meridional discs for P and Q. However, for a contraction we want to use caps which are disjoint from $P \cup Q$. Tube the current caps of Σ, namely meridional discs to P and Q, into the spheres $P^\perp$ and $Q^\perp$ respectively. Since the latter spheres are framed, the resulting caps still have the correct framing on the boundary. See Figure 12. The resulting caps lie in $M \setminus (P \cup Q)$, so the (immersed) sphere g produced by contraction does as well. Since each cap was used twice, the self-intersection number $\mu(g) = 0$. Since Σ has trivial normal bundle, and we contracted using correctly framed caps, $g(S^2)$ also have a trivial normal bundle. Finally, since we produced the sphere g by contracting a Clifford torus, it satisfies $\lambda(\Delta, g) = 1$.

Thus, the disc Δ and the algebraically dual sphere g in the manifold $M_{1/2} \setminus (P \cup Q)$ satisfy the requirements of the disc embedding Theorem 3.2. Use the resulting framed, flat, embedded Whitney disc $\overline{\Delta}$ in $M_{1/2} \setminus (P \cup Q)$ for the extraneous pair of intersections between P and Q to perform the Whitney trick on Q. In particular, this is a topological isotopy of Q. In the corresponding (topological) handle decomposition for W, the 2- and 3-handles can be cancelled, and we see that W is homeomorphic to the product $M_0 \times [0, 1]$, as claimed. $\square$

5. Outline of proof of the disc embedding theorem

In this section we give an extremely sketchy outline of the proof of the preliminary version of the disc embedding theorem stated before. The general version is proved in [Cas86, Fre82, FQ90], with an amendment in [PRT20] related to geometrically dual spheres. For a unified treatment with many details and pictures see [BKK+21]. We only outline the proof of the preliminary version of the theorem.

Broadly speaking, the proof has two main steps, as we now describe. We begin with a smooth, oriented, simply connected 4-manifold M, as well as a continuous map

$$\begin{array}{ccc} D^2 & \xrightarrow{f} & M \\ \uparrow & & \uparrow \\ \partial D^2 & \hookrightarrow & \partial M, \end{array}$$

where $f|_{\partial D^2}$ is a flat embedding, and an immersion $g\colon S^2 \looparrowright M$ such that the algebraic intersection number $\lambda(f,g)=1$, the algebraic self-intersection number $\mu(g)=0$, and $g(S^2)$ has trivial normal bundle.

The first step of the proof is the constructive step, and replaces a neighbourhood of $f(D^2)$ with a 4-dimensional object called a *skyscraper*. We will not precisely define what a skyscraper is, for now, restricting ourselves to the remark that a skyscraper is meant to be an approximation of a 2-handle. In particular, it has a prescribed attaching region, i.e. a subset of its boundary which is parametrised as a solid torus. Constructing a skyscraper uses techniques essentially due to Casson [Cas86], although his constructions were specifically in simply connected ambient 4-manifolds, and are called *Casson handles*. We prefer to talk about skyscrapers instead, since they apply in more general settings. The first step of the proof shows that there is a skyscraper in M, whose attaching region agrees with the framed boundary of $f(D^2)$.

The second step of the proof is the detection step, and comprises the key insight of Freedman, namely that every skyscraper is homeomorphic to $D^2 \times D^2$, relative to the attaching region. In other words, there is a homeomorphism from the skyscraper to $D^2 \times D^2$, which preserves the parametrisation of the boundary in a precise way. The core of this $D^2 \times D^2$ provides the embedding of a disc needed in the disc embedding theorem, and we see it is flat since it comes equipped with a product neighbourhood.

5.1. The constructive step of the proof.

Beginning with f and g, first perturb f to be an immersion with transverse self-intersections, and then tube every self-intersection into the unpaired intersection point between f and g, along an arc on f. I.e. take the ambient connected sum of $f(D^2)$ with many parallel pushoffs of $g(S^2)$. We still call the result f. Note that now all the intersections of f arise as parallel copies of intersections of g, and are therefore algebraically cancelling (this uses that $g(S^2)$ has trivial normal bundle). We have created more intersections between f and g in this process, as parallel pushoffs of the old intersections between f and g, but these also arise as algebraically cancelling pairs. At this point, we have arranged that $\mu(f)=0$, while preserving the boundary of f, and that $\lambda(f,g)=1$.

Next we perform a regular homotopy of f and of g, to arrange that f and g intersect geometrically in a single point. While this may seem counter intuitive at first (e.g. you might be wondering why we did not simply do this for the P and Q in the proof of Freedman's h-cobordism theorem), the key point is that the number of geometric self-intersections of f and of g may increase in this process. Let W be an immersed Whitney disc in M, pairing extraneous pairs of intersections between f and g (since $\lambda(f,g)=1$, all but one of the intersections of f and g can be paired up). After boundary twisting, we assume that W is framed. Push every intersection of f with the interior of W towards f along an arc on W, until the intersection is removed, by paying the price of two (algebraically cancelling) self-intersections of f. Similarly, push all intersections of g with the interior of W towards g along an arc on W. Make sure all the arcs used for pushing are disjoint. At the end of this process, the interior of W lies in the complement of f and g, so perform an (immersed) Whitney move on f along W. Still call the result f. This process removes one of the extraneous pairs of f, g intersections. Continue with the other pairs, until only the unpaired intersection point remains. At this point, we have that $\mu(f) = 0$ (since the new self-intersections arose as cancelling pairs), and $f \pitchfork g$ a single point.

To be continued

5.2. **The detection step of the proof.** We will barely scratch the surface of this step of the proof. Below we state the main theorem.

Theorem 5.1 (Skyscrapers are standard [Fre82, FQ90], see also [BKK+21]). *Every skyscraper is homeomorphic to $D^2 \times D^2$, relative to the attaching region.*

The proof requires understanding of *Kirby diagrams* and *decomposition space theory*. See the Piccirillo mini-course for much more about Kirby diagrams. Decomposition space theory is a beautiful, if somewhat outmoded, branch of topology, which seeks to address the following question.

Question 5.2. Let X be a metric space and $\mathcal{A} = \{A_1, A_2, \dots\}$ a disjoint collection of subsets of X. Let ${}^{X}\!/_{\mathcal{A}}$ denote the quotient of X by $\mathcal{A}$, where we identify each element of $\mathcal{A}$ to an individual point, i.e. a different point for each A_i. When is ${}^{X}\!/_{\mathcal{A}} \approx X$?

Other notable successes of decomposition space theory include Brown's proof of the Schoenflies theorem [Bro60] and Cannon's proof of the double suspension theorem [Can79] (see also [Edw06]).

Exercises for Lecture 2

Introductory problems.

Exercise 5.1. Let P and Q denote the xy- and zt-planes in $\mathbb{R}^4$ respectively, all three with their standard orientations. Let S denote the sphere of unit radius in $\mathbb{R}^4$, centred at the origin and orientation induced by the outward pointing normal. Show that $S \cap (P \cup Q)$ is the Hopf link. Which Hopf link is it?

Exercise 5.2. Let P and Q denote framed, embedded 2-spheres in a closed, smooth, oriented, simply connected 4-manifold $M_{1/2}$. Assume there are framed, immersed spheres $P^\perp$, $Q^\perp \subseteq M$ with $P^\perp \pitchfork P$ a point and $P^\perp \pitchfork Q = \emptyset$, as well as $Q^\perp \pitchfork Q$ a point and $Q^\perp \pitchfork P = \emptyset$. Show that $\pi_1(M_{1/2} \setminus (P \cup Q))$ is trivial. Which of the conditions in the hypotheses were needed in the argument?

Moderate problems.

Exercise 5.3 ((Equivariant) intersection number)**.** Let M be a connected, oriented, smooth 4-manifold with basepoint $m \in M$. Let $f\colon S^2 \looparrowright M$ and $g\colon S^2 \looparrowright M$ be smooth immersions. Fix an orientation on S^2 and a point $s \in S^2$. Assume that $f(S^2)$ and $g(S^2)$ intersect transversely. Let v_f and v_g be paths in M joining m to $f(s)$ and $g(s)$ respectively; these are called *whiskers* for f and g respectively. Define the following sum

$$\lambda(f,g) := \sum_{p \in f(S^2) \pitchfork g(S^2)} \varepsilon(p)\alpha(p),$$

where

- γ_f^p is the image of a simple path in S^2 from s to a point in $f^{-1}(p)$ and γ_g^p is the image of a simple path in S^2 from s to a point in $g^{-1}(p)$;
- $\varepsilon(p) \in \{\pm 1\}$ is the sign of the intersection point p (recall that both S^2 and M are oriented);
- $\alpha(p)$ is the element of $\pi_1(M,m)$ given by the concatenation $v_f \gamma_f^p (\gamma_g^p)^{-1} v_g^{-1}$.

Prove that

(i) $\lambda(f,g)$ does not depend on the choice of γ_f^p and γ_g^p.
(ii) if $\lambda(f,g) = 0$, then the points of $f(S^2) \pitchfork g(S^2)$ are paired up by maps of Whitney discs, i.e. the points can be paired by $\{p_1, q_1, p_2, q_2, \dots, p_n, q_n\}$ for some n, so that for each i, there exist arcs $w_i \subseteq f(S^2)$ and $w_i' \subseteq g(S^2)$, with endpoints p_i and q_i, such that that the concatenation $w_i' \cdot w_i^{-1}$ is freely null-homotopic in M.

Exercise 5.4. Formulate a version of Freedman's h-cobordism theorem for 4-manifolds with boundary. Outline a proof using relative handle decompositions.

Let C be a smooth, compact, contractible 4-manifold, i.e. a *cork*. Prove that every diffeomorphism $f\colon \partial C \to \partial C$ extends over C.

Why did we need C to be smooth and f to be a diffeomorphism (rather than just a homeomorphism)?

Challenge problems.

Exercise 5.5. Upgrade the disc embedding theorem to the following sphere embedding theorem:

Theorem 5.3 (Sphere embedding theorem, preliminary version)**.** *Let M be a smooth 4-manifold and $F\colon S^2 \looparrowright M$ a smooth immersion with only isolated double point singularities and $\mu(F) = 0$, i.e. the signed sum of the self-intersections is zero. Suppose there exists another smooth immersion $G\colon S^2 \looparrowright M$ so that $G(S^2)$ has trivial normal bundle, $F(S^2)$ and $G(S^2)$ intersect transversely, and $\lambda(F, G) = 1$, i.e. the signed sum of intersections between F and G is 1.*

Then F is homotopic to a flat embedding $\overline{F}\colon S^2 \hookrightarrow M$.

Use the following steps:

(1) Make f and g geometrically dual, i.e. up to adding more (algebraically cancelling) self-intersections of f and of g, assume that $f(S^2)$ and $g(S^2)$ intersect precisely once.
(2) Apply the disc embedding theorem to Whitney discs pairing the intersections of $f(S^2)$. Why can we get these discs to lie with interior in $M \setminus f(S^2)$?
(3) Use Clifford tori to produce algebraically dual spheres for the application of the disc embedding theorem.

Exercise 5.6. Use the sphere embedding theorem to construct a closed, topological 4-manifold with intersection form $E8 \oplus E8 \oplus 2H$, where H is the hyperbolic matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

The $K3$ surface is a good place to start, recall that it has intersection form $E8 \oplus E8 \oplus 3H$. Your goal is to do surgery to remove the extra H summand. What would be need to ensure the result of surgery is simply connected? Does there exist a closed, *smooth* 4-manifold with this intersection form?

6. Two paths to the fully topological 4-dimensional Poincaré conjecture

A special case of Theorem 1.1 states that every smooth homotopy 4-sphere M is homeomorphic to S^4. To see this note that $\pi_1(M) = 1 = \pi_1(S^4)$ and $H_2(M;\mathbb{Z}) = H_2(S^4;\mathbb{Z}) = 0$ so there is no intersection form.

As a reminder, the proof of Theorem 1.1 had two steps. First we applied a theorem of Wall (Theorem 2.8) to see that M and S^4 cobound a smooth, compact h-cobordism W. Then we applied Freedman's h-cobordism theorem (Theorem 4.1), whose proof was inspired by Smale's high dimensional version (Theorem 2.2), to see that the h-cobordism $(W; M, S^4)$ is homeomorphic to $S^4 \times [0,1]$. The homeomorphism restricted to M shows that $M \approx S^4$.

In all of this, we had to start by assuming that M is smooth. Our next aim is to remove this hypothesis, i.e. we will show the following theorem.

Theorem 6.1. *Let M be a topological manifold such that M is homotopy equivalent to S^4. Then M is homeomorphic to S^4.*

Observe that the above has the corollary that every homotopy 4-sphere admits a smooth structure: pull the standard smooth structure on S^4 back along the homeomorphism $M \to S^4$. However, this is not known *a priori*, so we cannot directly apply smooth techniques to the proof.

We have the two following proof strategies.

(1) Our first option is to build a topological h-cobordism from M to S^4, and then prove a topological compact h-cobordism theorem for 5-dimensional h-cobordisms. The latter is a result of Quinn which we will discuss. We will also discuss the former, but it will be based on more classical tools in topology (i.e. no Freedman-Quinn machinery needed).

(2) Freedman [Fre82] in fact followed a different strategy, in which he used smooth results as much as possible. This was partly motivated by the fact that the topological tools necessary to follow the first strategy did not yet exist, and Freedman wanted to prove the Poincaré conjecture as quickly as possible, in particular with time to spare before his 40th birthday.

    Specifically, he considered $M \setminus \{\mathrm{pt}\}$, which is a contractible open topological manifold. By results of Lashof from the 1970s [Las70a, Las70b, Las70c, Las71] we have that $M \setminus \{\mathrm{pt}\}$ is smoothable. One then sees that $M \setminus \{\mathrm{pt}\}$ and $\mathbb{R}^4$ (with its standard smooth structure) are smoothly *properly* h-cobordant (see Exercise 7.2). Then Freedman proved a noncompact proper h-cobordism theorem, showing that smooth, simply connected, 5-dimensional proper h-cobordisms that are in addition *simply connected at infinity*, are (proper) homeomorphic to products.

    As a consequence of the above, one sees that $M \setminus \{\mathrm{pt}\}$ and $\mathbb{R}^4$ are homeomorphic. Extending over one point compactifications, we see that M and S^4 are homeomorphic.

We will discuss the second approach further in a subsequent lecture. For now we focus on the first approach, because along the way we will establish many tools that are useful in other contexts.

6.1. **The first approach.**

6.1.1. *An analogue of Wall's theorem.* Let M^4 be a topological manifold. We will now show that if $M \simeq S^4$, then M and S^4 are topologically h-cobordant.

Definition 6.2. Let X be a topological space. A closed subspace $A \subseteq X$ is *1-locally coconnected (1-LCC)*, if for all $a \in A$ and for every neighbourhood $U \ni a$ there exists a neighbourhood $V \subseteq U$ such that $\pi_1(V \setminus A) \to \pi_1(U \setminus A)$ is the trivial map, for every choice of basepoint.

For example, for a point $x \in M^n$ in a manifold we have arbitrarily small neighbourhoods of x homeomorphic to $\mathbb{R}^n$, so $x \in M$ is 1-LCC provided $n \geq 3$.

Consider the homotopy sphere M. The cone on M is by definition the space $\operatorname{cone}(M) := M \times [0,1] \big/ M \times \{1\}$. One can show that $\operatorname{cone}(M)$ is an *ANR homology 5-manifold*; by definition, this means that $\operatorname{cone}(M)$ is an absolute neighbourhood retract and a homology 5-manifold. Briefly, a space X is said to be an absolute neighbourhood retract (ANR) if for every embedding of X as a closed subset $C(X)$ in a metrisable space Y, there is a neighbourhood $U(X)$ of $C(X)$ in Y and a retraction $U(X) \to C(X)$. Topological manifolds are ANRs. A *homology n-manifold* is a space Y satisfying

$$H_k(Y, Y \setminus \{x\}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n. \end{cases}$$

for every $x \in Y$. Note that topological n-manifolds are homology n-manifolds.

The only potentially non-manifold point of $\operatorname{cone}(M)$ is the cone point v. We observe that v is 1-LCC: for any neighbourhood U of v we can take $V \supseteq U$ as a "smaller cone", so $V \setminus v \simeq M$, which is simply connected.

By a theorem of Bryant and Lacher [BL78], the space $\operatorname{cone}(M)$ is then a topological 5-manifold. We can then remove an open 4-ball neighbourhood to obtain a topological h-cobordism from M to S^4. Once we have proven the fully topological 5-dimensional h-cobordism theorem, we can apply this to see that $M \approx S^4$. The Bryant-Lacher theorem is proved using techniques of decomposition space theory, which we mentioned earlier in Section 5. In particular, these techniques do not respect smooth structures.

6.1.2. *A fully topological h-cobordism theorem.* We begin with some definitions.

Definition 6.3. A k-dimensional (topological) submanifold $\Sigma \subseteq M^n$ is *locally flat* if every point $p \in \Sigma$ has a neighbourhood $U \supseteq \Sigma$ such that $(U, U \cap \Sigma) \approx (\mathbb{R}^n, \mathbb{R}^k)$.

Two (topological) submanifolds $\Sigma_1^k, \Sigma_2^l \subseteq M^n$ intersect *transversely* at a point p if there is a neighbourhood U of p such that

$$(U, U \cap \Sigma_1, U \cap \Sigma_2) \approx (\mathbb{R}^n, \mathbb{R}^k \times \{(0, \ldots, 0)\}, \{(0, \ldots, 0)\} \times \mathbb{R}^l).$$

In other words, within the neighbourhood U, Σ_1 and Σ_2 appear, up to homeomorphism, as transverse linear subspaces.

A useful exercise at this point is to return to the proof of Freedman's h-cobordism theorem (Theorem 4.1) and consider precisely where and how we used that the ambient manifold is smooth. We assert that only five ingredients were necessary, and we state below their topological counterparts, which are all we need in order to carry out the identical proof strategy in the fully topological setup.

(1) 5-dimensional topological manifolds admit topological handle decompositions (see Remark 2.4)
(2) Topological transversality: if Σ_1, Σ_2 are locally flat submanifolds of a topological 4-manifold then there exists a topological isotopy from Σ_1 to Σ_1' such that Σ_1' and Σ_2 intersect transversely.

(3) Every locally flat submanifold of a topological 4-manifold has a normal bundle, unique up to isotopy.
(4) The *immersion lemma*: Every continuous map $f\colon F \to M$, where F is a surface and M a topological 4-manifold is homotopic to a *topological immersion*, i.e. a map which is locally a locally flat embedding, where the self-intersections are transverse.
(5) The fully topological disc embedding theorem (preliminary version): Let M be a topological, oriented, simply connected 4-manifold. Suppose we have a map

$$\begin{array}{ccc} D^2 & \xrightarrow{f} & M \\ \uparrow & & \uparrow \\ \partial D^2 & \hookrightarrow & \partial M, \end{array}$$

where $f|_{\partial D^2}$ is a flat embedding. Suppose further that there is a topological immersion $g\colon S^2 \looparrowright M$ such that the algebraic intersection number $\lambda(f,g)=1$, the algebraic self-intersection number $\mu(g)=0$, and $g(S^2)$ has trivial normal bundle. Then there exists a flat embedding $\overline{f}\colon D^2 \hookrightarrow M$ with $\overline{f}|_{\partial D^2} = f|_{\partial D^2}$, and inducing the same framing on the boundary.

To summarise, once the above tools are available. we can repeat the proof of Theorem 4.1 in the topological category. As a reminder, ingredient 1 provides a handle decomposition for a compact, topological h-cobordism W^5 between M_0 and M_1, which by topological transversality we can assume consists of only 2- and 3-handles. Then we consider the middle level $M_{1/2}$, obtained after attaching the 2-handles. For simplicity like before we assume there is a single 2-handle and a single 3-handle, with belt sphere and attaching sphere P and Q, respectively, both in $M_{1/2}$. By topological transversality, we assume they intersect transversely. Using normal bundles for submanifolds, we construct immersed spheres $P^\perp$ and $Q_\perp$ such that $P^\perp \pitchfork P$ a point and $P^\perp \pitchfork Q = \emptyset$, as well as $Q^\perp \pitchfork Q$ a point and $Q^\perp \pitchfork P = \emptyset$. Since $M_{1/2}$ is simply connected, we assume that the extraneous pairs of intersections between P and Q are paired by maps of (framed) Whitney discs, which we assume are immersed by the immersion lemma. We construct algebraically dual spheres the Whitney discs using Clifford tori as before; this process again involves taking parallel pushoffs, and therefore uses the existence of nice tubular neighbourhoods in the background. Finally, we use the fully topological disc embedding theorem to upgrade the framed, immersed Whitney discs to embedded Whitney discs. (We only stated the theorem for the case of a single disc, but a more general version exists.) Performing the Whitney move on Q along these embedded discs is an isotopy, at the end of which P and Q intersect at a single point. Then we know that the corresponding 2- and 3-handle cancel, so that W is homeomorphic to $M_0 \times [0,1]$.

Remark 6.4. We do not recap the proof of the disc embedding theorem here, but indeed all we need to upgrade the proof from Section 5 is topological transversality, the existence of normal bundles for locally flat submanifolds, and the immersion lemma. These latter three facts, as well as the existence of handle decompositions for 5-manifolds, were proved by Quinn [Qui82]. A warning is in order: a key ingredient in his proof is the smooth-to-topological version of the disc embedding theorem (Theorem 3.2), or rather, more precisely, the fact that skyscrapers are homeomorphic to 2-handles (Theorem 5.1).

In the subsequent sections, we will indicate the proofs of the above list of ingredients necessary to prove the fully topological, compact h-cobordism theorem.

7. Smoothing noncompact 4-manifolds

Our goal in this section is to prove the following key result of Quinn [Qui82]. It will directly imply the immersion lemma, and the methods we develop in the proof will help us to prove other fundamental tools in topological 4-manifolds.

Theorem 7.1 (Quinn [Qui82])**.** *Every connected noncompact 4-manifold is smoothable.*

It follows immediately that every compact, connected 4-manifold is smoothable in the complement of a point.

The upcoming proof follows a general framework, which we introduce next. The following notion comes from [KS77, Essay I, Section 3].

Definition 7.2. Let V be a smooth n-manifold. A *handle smoothing problem* is a topological embedding

$$h\colon B^k \times \mathbb{R}^{n-k} \hookrightarrow V^n$$

such that h is smooth in a neighbourhood of $\partial B^k \times \mathbb{R}^{n-k}$.

We say h can be *solved (on $B^k \times B^{n-k}$)* if there exists a topological isotopy $h_t\colon B^k \times \mathbb{R}^{n-k} \hookrightarrow V$ from $h_0 = h$ to $h_1 = h'$ such that $h'|_{B^k\times B^{n-k}}$ is smooth and h_t is fixed on a neighbourhood of $\partial B^k \times \mathbb{R}^{n-k}$ and outside a compact neighbourhood of the core $B^k \times \{0\}$.

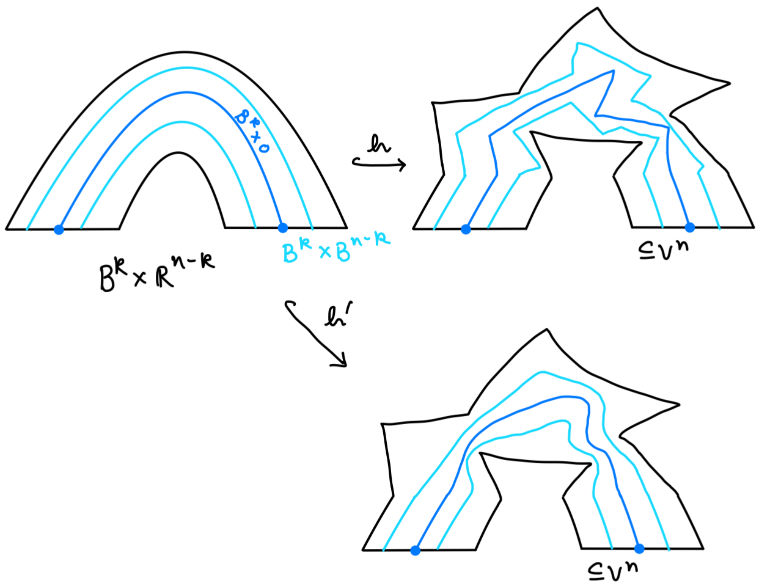


Figure 13. Handle smoothing

Remark 7.3. There is an analogue notion in the case of piecewise-linear (PL) manifolds, called a *handle straightening problem*, where we ask whether a topological embedding of an open handle can be isotoped to be a piecewise-linear embedding on its core.

For $n \leq 3$ handle smoothing problems can always be solved. This was shown using the torus trick by Hatcher [Hat13] in dimension 2 and Hamitlon [Ham76] in dimension 3 (Hamilton in fact showed that in dimension 3 handle straightening problems can be solved, but deep results in 3-manifold topology show that every handle smoothing problem can also be solved).

For $n \geq 5$ and $k \neq 3$, handle straightening problems can always be solved. The missing $k = 3$ case is related to the Kirby-Siebenmann invariant. In general high-dimensional handle smoothing problems cannot be solved: otherwise the smooth Poincar'e conjecture would hold in all dimensions. However for $n \geq 5$, handle straightening and smoothing problems that are *concordant* to a solved problem can be solved, which turns out to be extremely useful. These can all be proved using Kirby's celebrated *torus trick*. The first instance of this was in the $k = 0, n \geq 5$ case by Kirby [Kir69]. The other high-dimensional cases were done in [KS77, Essay I]. Handle straightening and smoothing is a key ingredient in Kirby-Siebenmann's proof that concordance implies isotopy for piecewise-linear and smooth structures in high dimensions, from which they derived the *product structure theorem*, a key ingredient in establishing fundamental tools (like topological transversality and the existence of topological handle decompositions) in high dimensions. The product structure theorem, together with the fact that the groups of homotopy spheres are finite, can also be used to show that compact manifolds of dimension ≥ 5 have at most finitely many smooth structures.

To motivate the upcoming proof of Theorem 7.1, we show that smoothing handles provides smooth structures on manifolds.

Theorem 7.4. *Fix n and assume all handle smoothing problems for n-dimensional handles of every index are solvable. Then every topological n-manifold with empty boundary admits a smooth structure.*

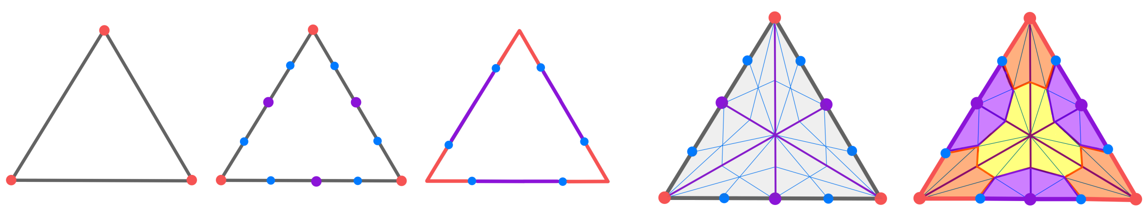
Note that by [Hat13, Ham76], this shows that all topological n-manifolds with dimension ≤ 3 admit smooth structures. See also Exercise 7.4 for other applications of handle smoothing.

Proof. Let M be a topological n-manifold. Let $\{h_i : \mathbb{R}^n \to M^n\}$ be a locally finite, countable collection of coordinate charts. This means that every compact set in M intersects at most finitely many charts. Define $U_j := \bigcup_{i \leq j} h_i(\mathbb{R}^n)$. We will smooth M by induction, which will suffice despite the possibility that we have infinitely many charts since the collection is locally finite. U_1 admits a smooth structure as the homeomorphic image of $\mathbb{R}^n$. Suppose we have a smooth structure on U_{j-1}. Define $W := h_j^{-1}(U_{j-1})$. Note that W is an open subset of $\mathbb{R}^n$.

There is a triangulation of W consisting of simplices whose size decreases to zero as we approach the frontier. To see this, start with a cubulation of $\mathbb{R}^n$ into unit cubes. Subdivide each cube with nonempty, proper intersection with W, and iterate to infinity. Subdivide the resulting cubulation of W into simplices to get a triangulation. A standard process produces a handle decomposition from a triangulation. Explicitly, for a k-simplex σ in a triangulation T of a manifold N, we obtain a handle of index k given by

$$\mathrm{St}(\widehat{\sigma}) \subseteq T''$$

where T'' is the second barycentric subdivision of T, $\widehat{\sigma}$ is the barycentre of σ, and St denotes the star. See Figure 14 for an example and [Hud69, p. 233] for further details.



(A) Circle

(B) 2-simplex

FIGURE 14. Construction of a handlebody decomposition from a triangulation. 0-handles are coloured orange, 1-handles are purple, and the 2-handle is yellow.

By the construction of the handle decomposition from the triangulation, we see that each handle $B^k \times B^{n-k}$ extends to a neighbourhood $B^k \times \mathbb{R}^{n-k}$. We now solve a series of handle smoothing problems, iterating on the index of the handles, beginning with the 0-handles. At each stage, the handle smoothing problem is $B^k \times \mathbb{R}^{n-k} \hookrightarrow U_{j-1}$ and we will solve it on $B^k \times 2B^{n-k} \subseteq B^k \times \mathbb{R}^{n-k}$ (i.e. on a slight thickening of $B^k \times B^{n-k}$ in the cocore direction). The purpose of using the ball of radius 2 is to smooth a neighbourhood of the attaching region of handles of the next higher index, in order to continue the iteration. Combining all the isotopies so far achieved, and extending by the constant isotopy outside of W, we obtain an isotopy from h_j so some new chart h_j'. (Since the simplices have size decreasing to zero as we approach the frontier of W, extending by the constant isotopy is indeed continuous.) The end result h_j' of the isotopy has the same image in M as h_j, but since we made h_j' smooth on W, we now obtain a smooth structure on U_j, compatible with that on U_{j-1}, as desired. □

We saw earlier that even though high-dimensional handle smoothing problems cannot always be solved, even partial solutions can have powerful consequences. The same principle applies here; in particular, Quinn showed the following partial solution for handle smoothing in ambient dimension four. Note that the result of this theorem is sharp, as indicated in [FQ90, Section 8.5].

Theorem 7.5 (Quinn)**.** *Let V^4 be a smooth 4-manifold, and let $h\colon B^k \times \mathbb{R}^{4-k} \hookrightarrow V$ be an embedding such that $h|_{\mathrm{nbd}(\partial B^k \times \mathbb{R}^{4-k})}$ is smooth.*

- *(k=0,1) Then h can be solved on $B^k \times B^{4-k}$.*
- *(k=2) Then h can be* solved *on W, where W is a neighbourhood of $B^k \times 0$ after either:*
  (1) topological ambient isotopy rel. $\partial B^k \times \mathbb{R}^{4-k}$ and rel. infinity; or
  (2) a smooth regular homotopy,
  namely $h_t\colon B^k \times B^{n-k} \hookrightarrow V$ rel $\partial B^k \times \mathbb{R}^{4-k}$ and infinity, with $h_0 = h$ and $h_1|_W$ smooth.

In other words, one cannot immediately smooth 2-handles, but after doing something to the core (either a topological ambient isotopy or a smooth regular homotopy), it is possible to get a solution on a neighbourhood of this new subset.

Note that the smooth regular homotopy part of the $k = 2$ case above implies that every locally flat slice disc in D^4 can be arbitrarily closely approximated by a smooth immersed disc; indeed the smooth immersed disc is produced from the locally flat slice disc by (small) finger moves.

In addition to Theorem 7.1, Theorem 7.4 has the following powerful corollary.

Corollary 7.6 (4-dimensional annulus theorem)**.** *If $f\colon D^4 \hookrightarrow \mathrm{Int}(D^4)$ is a collared topological embedding, then $D^4 \setminus f(\mathrm{Int}\, D^4) \approx S^3 \times [0,1]$.*

The higher-dimensional analogue of the annulus theorem was proven by Kirby using the torus trick in [Kir69]. Quinn's result also completed the proof of the stable homeomorphism conjecture of Brown and Gluck [BG64]. Here is an important corollary of the annulus theorem.

Corollary 7.7. *Connected sum of connected, (oriented/nonorientable) topological 4-manifolds is well defined.*

For the above two statements, see Exercise 8.2 and Exercise 8.1.

We now prove Theorem 7.1.

Proof. Let M^4 be a connected noncompact 4-manifold. First we give the proof modulo the following claim.

Claim. *There exists a discrete set $S = \{s_\alpha\} \subseteq M$ with $M \setminus S$ smoothable.*

There exists a locally finite collection of disjoint proper flat rays $r_\alpha\colon [0,\infty) \to M$ with $r_\alpha(0) = s_\alpha$. One way to build these rays is to consider an exhaustion by compact sets and create the rays inductively. Details are omitted.

Then $M \setminus \cup_\alpha r_\alpha([0,\infty))$ is an open submanifold of $M \setminus S$, which is smoothable by the first claim. Therefore, $M \setminus \cup_\alpha r_\alpha([0,\infty))$ is smoothable. However, we also observe that $M \setminus \cup_\alpha r_\alpha[0,\infty)$ is homeomorphic to M, and therefore M is smoothable. This completes the proof of Theorem 7.1 modulo the proof of the claim, which we now give.

Proof of the claim. Let $\{h_i\colon \mathbb{R}^4 \to M\}$ be a countable, locally finite collection of coordinate charts for M and let

$$U_j := \bigcup_{i \leq j} h_i(\mathbb{R}^4).$$

Assume by induction that there exists a discrete set $S_{j-1} \subseteq U_{j-1}$ so that $U_{j-1} \setminus S_{j-1}$ is smooth. To prove the induction step we will add (discrete) points in $h_j(\mathbb{R}^4)$ to S_{j-1} to form the set $S_j \subseteq U_j$ so that $U_j \setminus S_j$ is smooth.

Choose a bicollared codimension one smooth submanifold $V \subseteq \mathbb{R}^4$ so that $h_j(V)$ separates $U_{j-1} \setminus h_j(\mathbb{R}^4)$ and $h_j(\mathbb{R}^4) \setminus U_{j-1}$.

Now apply Quinn's handle smoothing (Theorem 7.4) to $V \times (-1,1)$. Mor precisely, note that V is a 3-manifold – take a triangulation of V, and then consider the induced handle decomposition. Take the product of this handle decomposition with $(-1,1)$ to obtain a handle decomposition of $V \times (-1,1)$. Note that although $V \times (-1,1)$ is a 4-manifold, by our construction of the handle decomposition, we do not have any 4-handles. Using the handle smoothing theorem (the version with topological isotopy for 2-handles), we obtain an isotopy of h_j to h_j', fixed outside $h_j^{-1}(U_{j-1})$ with h_j' smooth on a neighbourhood of the 2-skeleton of V', where V' is obtained from V by a topological ambient isotopy of $\mathbb{R}^4$. Let W' denote this neighbourhood of V'. By choosing the isotopy and neighbourhoods small enough, and since S_{j-1} is discrete, we can ensure that W' does not intersect S_{j-1} and that V' still separates $U_{j-1} \setminus h_j(\mathbb{R}^4)$ and $h_j(\mathbb{R}^4) \setminus U_{j-1}$. Since V' is a 3-manifold, we also know that the complement of the 2-skeleton is a (discrete) collection of 3-balls $\{B_\ell\}$.

We will use the neighbourhood $h_j'(W')$ to patch together the smooth structures on U_{j-1} and $h_j'(\mathbb{R}^4)$. The latter has the smooth structure inherited from the domain $\mathbb{R}^4$. Here we know that $h_j'(W')$ is a smooth submanifold of $h_j'(\mathbb{R}^4)$, whereas we only have that the image under h_j' of the complement in W' of the (now thickened) balls $\{B_\ell\}$ is smooth in U_{j-1}. As a result we have a smooth structure on U_j away from $S_j := S_{j-1} \cup \{h_j'(\mathrm{centres}(B_\ell))\}$, which is discrete. □

□

Exercises for Lecture 3

Introductory problems.

Exercise 7.1. Let M be a closed n-manifold. Assume that M is simply connected and $H_*(M;\mathbb{Z}) \cong H_*(S^n;\mathbb{Z})$. Show that M is homotopy equivalent to S^n.

Moderate problems.

Exercise 7.2.

Definition 7.8. A map between spaces $f\colon X \to Y$ is called *proper* if the inverse image $f^{-1}(K)$ of every compact set $K \subseteq Y$ is compact in X. A proper map $f\colon X \to Y$ is said to be a *proper homotopy equivalence* if there is a proper map $g\colon Y \to X$ such that $f \circ g$ and $g \circ f$ are properly homotopic to Id_Y and Id_X respectively, meaning that the homotopies are proper maps. A *proper h-cobordism* W is a cobordism between manifolds M_0 and M_1, such that the inclusions $M_i \hookrightarrow W$ are proper homotopy equivalences.

Let M be a smooth 4-manifold with empty boundary which is properly homotopy equivalent to $\mathbb{R}^4$. Let $B \subseteq M$ denote the interior of a smooth ball in M. Show that $M \times [0,1) \cup B \times \{1\}$ is a smooth, proper h-cobordism between M and B.

Exercise 7.3. Look up the *trace embedding lemma*, that a knot is topologically (resp. smoothly) slice if and only if the 0-trace has a collared (resp. smooth) embedding in $\mathbb{R}^4$. Assume there exists a knot which is topologically slice but not smoothly slice. Use Theorem 7.1 to show that there is an exotic smooth structure on $\mathbb{R}^4$.

Challenge problems.

Exercise 7.4. Fix n and assume that handle smoothing problems for n-dimensional handles of every index are solvable.

Prove that every topological n-manifold, with potentially nonempty boundary, admits a smooth structure. Prove that every homeomorphism of smooth n-manifolds is isotopic to a diffeomorphism.

Health warning: One can prove anything with a false premise. Look up some counterexamples to the above statements.

8. Flowchart

We have seen that Quinn's handle straightening implies the annulus theorem in dimension four (Corollary 7.6 and Exercise 8.2) and that connected, noncompact 4-manifolds are smoothable (Theorem 7.1). The 4-dimensional annulus theorem implies that the connected sum of oriented 4-manifolds is well defined (Corollary 7.7 and Exercise 8.1). In other words, we have addressed the right hand side of the diagram below (Figure 15). The diagram indicates paths to proving some of the other fundamental tools in 4-manifold topology indicated in Section 6, the immersion lemma, the existence of normal bundles for locally flat submanifolds, and topological transversality. As one sees from the flowchart, the key additional result is the controlled h-cobordism theorem, which was proved by Quinn in [Qui82].

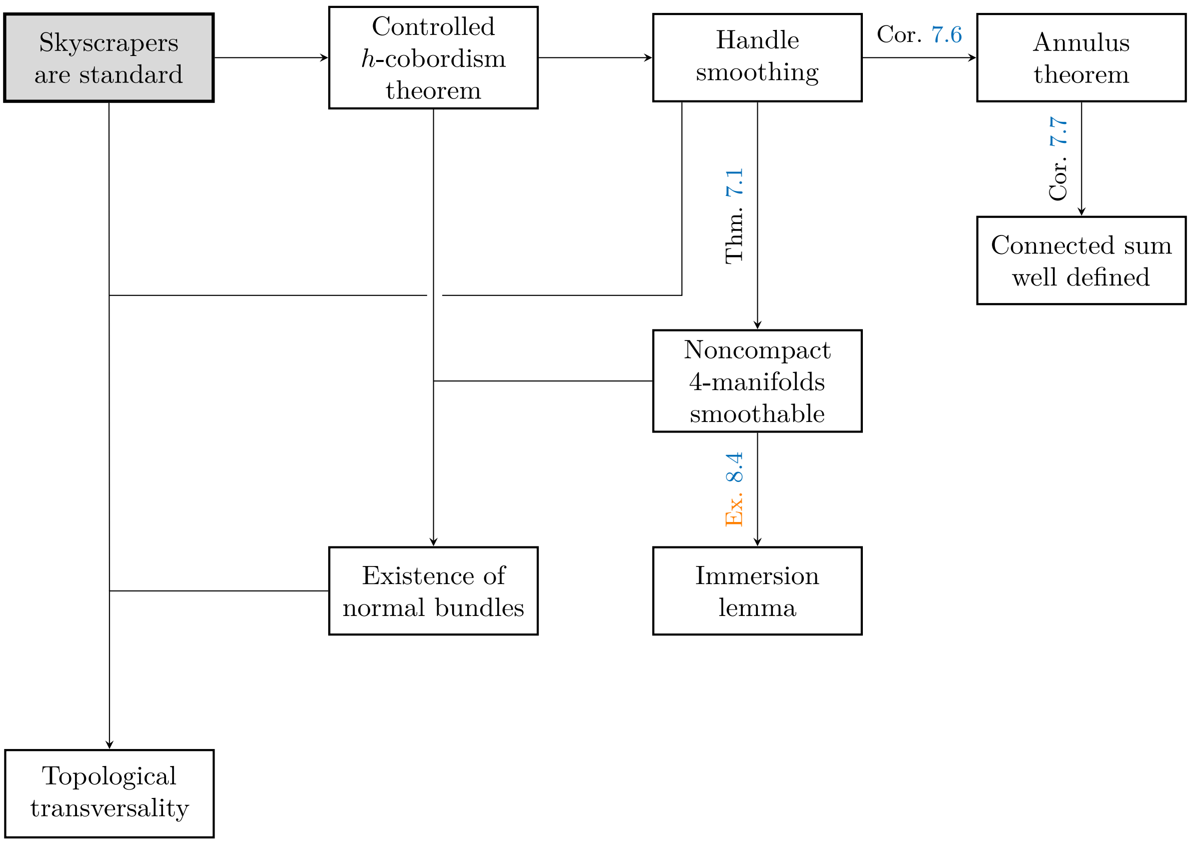


Figure 15. Logical dependence of some results discussed in the mini-course.

The full discussion of the proofs is beyond the scope of this mini-course. For now we limit ourselves to sketching the proof of the existence of normal bundles for locally flat submanifolds. The following is a special case of the result proved by Quinn [FQ90, Theorem 9.3A].

Theorem 8.1. *Let X be a topological space and $A \subseteq X$ closed. Moreover, assume the following:*

(1) X is ANR homology 4-manifold;
(2) $X \setminus A$ is a manifold;
(3) $A \times \mathbb{R}$ is a manifold;
(4) Either $\dim A \neq 2$ and A is 1-LCC, or $\dim A = 2$ and for all $a \in A$ and all $U \ni a$ there exists a neighbourhood $V \subseteq U$ such that $\operatorname{Im}(\pi_1(V \setminus A) \to \pi_1(U \setminus A)) \cong \mathbb{Z}$.

Then A has a normal bundle in X.

Remark 8.2. What does $\dim A$ mean? One can take it to mean one less than the dimension of $A \times \mathbb{R}$. However one could also apply dimension theory, which allows one to give a meaning to the dimension of spaces in more generality.

The above theorem implies that closed, locally flat submanifolds have normal bundles. To see this one must show that locally flat submanifolds have the correct local homotopy groups (Exercise 8.3). To extend to potentially non-closed submanifolds, one needs the relative version of the theorem [FQ90, Theorem 9.3A], which is then applied inductively.

Remark 8.3. As previously mentioned, fundamental tools in high-dimensional manifold topology were established by Kirby-Siebenmann [KS77], and in general Quinn's work shows that analogous tools are available in dimension four. The case of normal bundles is curious – not all locally flat submanifolds in high dimensions are known to have normal bundles – we only know this for certain codimensions. However, in dimension four there is no codimension restriction.

, which is the middle box at the bottom of the flowchart. It is implied by the noncompact smoothing theorem. To see this consider $X \times \mathbb{R}$ and remove a neighbourhood of $A \times \mathbb{R}$, then use the controlled h-cobordism theorem.

In fact, the controlled h-cobordism theorem implies handle smoothing. The controlled h-cobordism theorem follows from applying "Skyscrapers are standard", after setting up the controlled h-cobordism to have infinitely many skyscrapers of controlled size, all of which we want to replace by locally flat Whitney discs.

Theorem 8.4 (Freedman)**.** *Let Σ be an integral homology 3-sphere. Then $\Sigma = \partial C^4$ for a compact, contractible, topological 4-manifold C.*

Proof. Consider $\Sigma \times I$. Pick generators for $\pi_1(\Sigma)$, push them into interior of $\Sigma \times I$, and surger them: remove copies of $S^1 \times D^3$ and replace them with $D^2 \times S^2$. We get closed surfaces obtained as unions of Seifert surfaces and newly added discs $D^2 \times \{p\} \subseteq D^2 \times S^2$ obtained from the $D^2 \times S^2$ s added in the surgery. This new 4-manifold has trivial fundamental group, so homology classes are represented by immersed spheres with algebraic dual spheres from $\{q\} \times S^2 \subseteq D^2 \times S^2$. Apply the sphere embedding theorem to make them embedded, then surger them out. The resulting W is simply connected and has trivial second homology.

Now we stack countably many copies of W, and consider the one point compactification of this infinite stacking. (We keep one copy of Σ untouched.) The vertex ∞ added for the 1-point compactification is 1-LCC, and we check other properties of normal bundles theorem are satisfied. ∎

Exercises for Lecture 4

Introductory problems.

Exercise 8.1.

Theorem 8.5 (Palais [Pal60])**.** *Any two smooth orientation-preserving smooth embeddings of D^n in a connected oriented smooth n-manifold are smoothly equivalent, i.e. there is an orientation-preserving diffeomorphism of the ambient manifold taking one to the other.*

Use Palais's disc theorem to prove that connected sum of oriented, smooth n-manifolds is well defined.

Exercise 8.2 (4-dimensional annulus theorem)**.** Prove Corollary 7.6.

Exercise 8.3. We begin with some definitions (compare Definition 6.2).

Definition 8.6. Let $e\colon M^m \hookrightarrow N^n$ be an embedding. We say that e is *locally flat at* $x \in M$ (or at $e(x) \in N$) if there exists a neighbourhood U of $e(x)$ in N and a homeomorphism:

$$\begin{cases} h\colon U \to \mathbb{R}^n \text{ such that } h(U \cap e(M)) = \mathbb{R}^m \subseteq \mathbb{R}^n, & \text{if } x \in \operatorname{Int} M,\ e(x) \in \operatorname{Int} N, \\ h\colon U \to \mathbb{R}^n \text{ such that } h(U \cap e(M)) = \mathbb{R}^m_+ \subseteq \mathbb{R}^n, & \text{if } x \in \partial M,\ e(x) \in \operatorname{Int} N, \\ h\colon U \to \mathbb{R}^n_+ \text{ such that } h(U \cap e(M)) = \mathbb{R}^m_+ \subseteq \mathbb{R}^n_+, & \text{if } x \in \partial M,\ e(x) \in \partial N. \end{cases}$$

We say that e is *locally flat* if it is locally flat at each point.

Definition 8.7. Let $A \subseteq X$ be a closed subset of a topological space.

(1) We say that A is *k-locally co-connected at $a \in A$*, written k-LCC at a, if for every neighbourhood U of a there exists an open neighbourhood V with $a \in V \subseteq U$ such that any $S^k \to V \setminus A$ extends as

$$\begin{array}{ccc} S^k & \longrightarrow & V \setminus A \\ \downarrow & & \downarrow \\ D^{k+1} & \longrightarrow & U \setminus A \end{array}$$

In other words, $\pi_k(V \setminus A) \to \pi_k(U \setminus A)$ is trivial for every choice of basepoints for which this makes sense.

(2) We say that A has a 1-*abelian local group at* $a \in A$, written 1-alg, if for every neighbourhood U of a there exists an open neighbourhood V with $a \in V \subseteq U$ such that the inclusion induced homomorphism $\pi_1(V \setminus A) \to \pi_1(U \setminus A)$ has abelian image, for every choice of basepoints for which this makes sense.

(3) We say that A is *locally homotopically unknotted in X at* $a \in A$ if A is both 1-alg and k-LCC at a for every $k \neq 1$.

Suppose $M^m \subseteq N^n$ is locally flat. If U is as in the first case of Definition 8.6, show the following:

- If $n - m = 1$, then $\operatorname{Int} M$ is k-LCC for all $k \geq 1$ except $k = 0$.
- If $n - m = 2$, then $\operatorname{Int} M$ is locally homotopically unknotted in N at every point.
- If $n - m > 2$, then $\operatorname{Int} M$ is k-LCC for all $k \leq n - m - 2$.

If U is as in the second case of Definition 8.6, show the following: ∂M is k-LCC in $\operatorname{Int} M$ for all k in this case.

If U is as in the third case of Definition 8.6, show the following:

$$
\begin{aligned}
U \setminus M \cap U \approx (\mathbb{R}^1_+ \times \mathbb{R}^{n-1}) \setminus (\mathbb{R}^1_+ \times \mathbb{R}^{m-1}) &\approx \mathbb{R}^1_+ \times (\mathbb{R}^{n-1} \setminus \mathbb{R}^{m-1}) \\
&\approx \mathbb{R}^1_+ \times \mathbb{R}^{m-1} \times (\mathbb{R}^{n-m} \setminus \{0\}) \simeq S^{n-m-1},
\end{aligned}
$$

so ∂M is k-LCC in ∂N for all $k \leq n-m-2$ in this case.

Remark 8.8. The converse in the second case is also true: if $e \colon M \hookrightarrow N$ is an embedding, $n - m = 2$, and $\operatorname{Int} M$ is locally homotopically unknotted in N at every point, then e is locally flat. This is due to Chapman for dimension ≥ 5 [Cha79] and Quinn for dimension 4 [Qui82] (see also [FQ90, Theorem 9.3A, Lemma 9.3B]).

There are converses in the other codimensions as well, such as in [Čer73]. Indeed, these may be applied to certain generalisations of manifolds. See Theorem 8.1 and [DV09, Chap. 7, Chap. 8].

Moderate problems.

Exercise 8.4. Use Theorem 7.1 to prove the immersion lemma (Section 6).

9. Lecture 5

Corollary 9.1. *There is a topological 4-manifold* $*\mathbb{CP}^2$ *homeomorphic to* $\mathbb{CP}^2$ *but not diffeomorphic to it.*

Proof. We start with one 0-handle and one 2-handle, attached to any knot with nontrivial Arf invariant with framing +1. The boundary is an integer homology sphere, so we can close this up using a contractible 4-manifold from Theorem 8.4. The result is a simply connected closed 4-manifold with $H_2 \cong \mathbb{Z}$ and intersection form +1, so it is homeomorphic to $\mathbb{CP}^2$. It is not diffeomorphic since... □

Exercise 9.2. Find (smooth) 4-manifolds that are homotopy equivalent but are not homeomorphic.

Some other results...

9.1. **Application to knots.**

Theorem 9.3. *Alexander polynomial one knots are topologically slice.*

Proof. Start with the annulus $K \times [0, 1] \subseteq \Sigma \times [0, 1]$ and go through the steps in the proof of Theorem 8.4 to obtain a 4-manifold W. Stack them up, get a cone-like disk, then make flat... □

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